Local Topological Constraints on Berry Curvature in Spin–Orbit Coupled BECs

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Abstract

We establish a local topological obstruction to the simultaneous flattening of Berry curvature in spin–orbit-coupled Bose–Einstein condensates (SOC BECs), which remains valid even when the global Chern number vanishes. For a generic two-component SOC BEC, the extended parameter space is the total space M of a principal $U(1)_+ \times U(1)_-$ bundle over the Brillouin torus T^2_{BZ}. On M, we construct a Kaluza–Klein metric and a natural metric connection ∇^C whose torsion 3-form encodes the synthetic gauge fields. Under the physically relevant assumption of constant Berry curvatures, the harmonic part of this torsion defines a mixed cohomology class

$$[\omega] \in \big(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+})\big) \oplus \big(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_-})\big)$$

with mixed tensor rank $r = 1$. By adapting the *Pigazzini–Toda (PT) lower bound* to the Kaluza–Klein setting through exact pointwise curvature analysis, we demonstrate that the obstruction kernel $\mathcal{K}$ vanishes for the physical metric, yielding the cohomological invariant $r^\sharp = 1$. The exact curvature formula reveals a three-level irreducibility structure: (i) for the one-parameter deformation family g_ε interpolating between the product metric and the physical Kaluza–Klein metric, $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ holds at every point for all $\varepsilon \in (0,1)$; (ii) at the physical endpoint $\varepsilon = 1$, the natural Kaluza–Klein torsion—identified as the Bismut torsion—produces an exact cancellation of the off-diagonal curvature, which we characterise as a non-generic phenomenon: every other torsion representative of $[\omega]$ yields $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ on an open non-empty set; (iii) the Riemannian holonomy of the physical metric is itself irreducible, with $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}}) \geq 1$ at every point. These bounds prevent the complete gauging-away of Berry phases even in regimes with zero net topological charge. This work provides the first cohomological lower bound, based on the PT framework, certifying locally irremovable curvature in SOC BECs beyond the Chern-number paradigm, and identifies the Bismut cancellation as the unique obstruction to extending the bound from the deformation family to the natural Kaluza–Klein connection.

Keywords: Berry curvature; spin–orbit-coupled Bose–Einstein condensates; synthetic gauge fields; holonomy with torsion; mixed cohomology; Kaluza–Klein geometry.
MSC (2020): 53C29, 53C80, 81V45, 82B10, 51P05, 55S20, 57R19, 57R56, 58A12.

1 Introduction

The study of spin–orbit-coupled (SOC) Bose–Einstein condensates (BECs) stands at a rich confluence of differential geometry, quantum many-body physics, and synthetic gauge theory. This interdisciplinary effort is part of the broader pursuit of artificial gauge fields in engineered quantum systems, spanning ultracold atoms, photonics, and solid-state platforms [1].

Advances in generating artificial gauge fields with Raman lasers have made these systems a versatile laboratory for exploring Berry curvature, synthetic magnetic fields, and non-Abelian textures with unprecedented control [6, 7, 14]. Experimental realizations in laboratories such as NIST [9] have demonstrated the ability to engineer synthetic spin-orbit coupling in ultracold atomic gases, providing a controllable platform for investigating topological phenomena in quantum matter.

A persistent theoretical challenge is to identify *local* topological obstructions that constrain the Berry curvature even when global invariants—such as the first Chern number—vanish. While the Chern number $c_1 = \frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F$ quantizes the net flux through the Brillouin zone, its vanishing does not guarantee that the curvature F can be locally flattened everywhere; geometric obstructions may prevent the simultaneous annihilation of curvature components along independent directions in the parameter space. Understanding such local constraints is crucial for assessing the robustness of topological features in quantum systems and for designing protocols that exploit geometry beyond global topological charges.

This work addresses that challenge by bringing a recent geometric result into physical focus. In [12], a lower bound was established for the off-diagonal holonomy of metric connections with totally skew-symmetric torsion on product manifolds. The bound is expressed in terms of a *mixed* deRham cohomology class $[\omega] \in H^p(M_1) \otimes H^q(M_2)$ associated with the harmonic part of the torsion 3-form:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^C) \;\geq\; r^\sharp := \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}, \tag{1.1}$$

where $r = \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}})$ is the minimal number of simple tensors needed to represent the mixed cohomology class, $\mathcal{K}$ is an obstruction kernel encoding the intersection of this class with spaces of parallel forms, and $\mathfrak{hol}^{\mathrm{off}}(\nabla^C)$ denotes the off-diagonal part of the holonomy algebra that mixes the two factor manifolds. This bound is a topological invariant that persists under metric deformations preserving the parallel-form strata.

Methodological adaptation: From product to Kaluza–Klein geometry

While Theorem 5.2 of [12] is formally stated for Riemannian product metrics, the physical metric g_M in our SOC BEC model is a Kaluza–Klein metric induced by the synthetic gauge fields $A^{(\pm)}$. *We do not apply the theorem as a black box.* Instead, we extend its reach to the present geometry through a four-step strategy:

(i) *Deformation.* We construct a smooth one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ that interpolates between a Riemannian product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ at $\varepsilon = 0$ and the physical Kaluza–Klein metric $g_M = g_1$ at $\varepsilon = 1$ (Section4). This deformation is achieved by continuously introducing the gauge potentials $A^{(\pm)}$ into the fiber metric via the connection 1-forms $\Theta^{(\pm)}_\varepsilon = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$.

(ii) *Exact curvature analysis for $\varepsilon \in (0,1)$.* For every $\varepsilon > 0$, the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes (Theorem3.14), yielding a non-trivial reduced rank $r^\sharp_\varepsilon = 1$. By explicit computation in an adapted gauge (AppendixB), we derive the exact curvature formula $R^{C_\varepsilon}(e_1, e_3)\, e_3 = \frac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_1$, which is non-zero for all $\varepsilon \in (0,1)$, establishing $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ at every point $p \in M$.

(iii) *Bismut cancellation at the KK endpoint.* At $\varepsilon = 1$, the exact formula reveals that all off-diagonal curvature components vanish for the natural Kaluza–Klein torsion. We identify this as the *Bismut cancellation*: the Levi-Civita and contorsion contributions cancel exactly, rendering horizontal parallel transport block-diagonal. This phenomenon is specific to the Bismut torsion—the unique totally skew-symmetric torsion for which the horizontal distribution of the principal bundle is parallel along horizontal directions—and is non-generic among torsion representatives of $[\omega]$.

(iv) *Three-level irreducibility at the physical metric.* We establish the lower bound for the physical metric g_1 through three complementary results (Theorem5.1): every non-Bismut torsion representative of $[\omega]$ yields $\dim \mathfrak{hol}^{\text{off}} \geq 1$ on an open non-empty subset of M; the Riemannian holonomy of g_1 is itself irreducible, with $\dim \mathfrak{hol}^{\text{off}}(\nabla^{\text{LC}}) \geq 1$ at every point; and the Bismut torsion is characterised as the unique exception of infinite codimension in the space of torsion representatives. The triviality of the bound at $\varepsilon = 0$ (where $r_0^\sharp = 0$ due to complete absorption by parallel forms in the product geometry) highlights the essential role of the Kaluza–Klein coupling in generating irreducible topological constraints.

We take as the extended parameter space the smooth manifold

$$M = T^2_{\text{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-},$$

where T^2_{BZ} is the Brillouin torus parametrized by crystal momenta (k_x, k_y), and $S^1_{\phi_+}$, $S^1_{\phi_-}$ represent the global U(1) phase (associated with particle-number conservation) and the relative U(1) phase between the two spin components, respectively. Our analysis shows that under the physically motivated assumption of constant Berry curvatures $F^{(\pm)} = c^{(\pm)}\text{vol}_{\text{BZ}}$ (Assumption2.3), the mixed tensor rank is $r = 1$. The vanishing of the obstruction kernel for the Kaluza–Klein geometry (Theorem2.14) yields the cohomological invariant $r^\sharp = 1$. The exact curvature analysis then establishes a three-level irreducibility structure for the physical metric:

$$\begin{aligned}
&\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_\varepsilon}) \geq 1 && \text{for all } \varepsilon \in (0,1) \text{ at every } p \in M, \\
&\dim \mathfrak{hol}^{\text{off}}(\nabla^{C'}) \geq 1 && \text{for non-Bismut } T' \text{ on an open non-empty subset of } M, \\
&\dim \mathfrak{hol}^{\text{off}}(\nabla^{\text{LC}(1)}) \geq 1 && \text{at every } p \in M.
\end{aligned} \tag{1.2}$$

These bounds imply that momentum degrees of freedom are fundamentally "locked" to the phase sectors in a topologically robust manner: the Riemannian geometry of the physical Kaluza–Klein metric is irreducible with respect to the horizontal–vertical splitting, and this irreducibility persists at the torsion-connection level for every non-Bismut representative of $[\omega]$. The Bismut cancellation—the exact compensation of the Levi-Civita and contorsion terms at $\varepsilon = 1$—is a non-generic phenomenon of infinite codimension, broken by any physical perturbation that modifies the torsion representative within the same cohomology class. The mixing of the Brillouin zone with the phase directions cannot be removed by any smooth gauge transformation or metric deformation preserving the cohomology classes $[F^{(\pm)}]$, even when the total Chern number $c_1^{(+)} + c_1^{(-)}$ vanishes. This result reveals a *local* topological obstruction invisible to the Chern number and suggests new interferometric protocols to detect these geometric constraints through measurements of correlated Berry phases in the two U(1) sectors.

Organization of the paper

The paper is organized as follows. Section2 constructs the Kaluza–Klein metric and defines the metric connection with torsion encoding the synthetic gauge fields. Section3 analyzes the mixed cohomology structure, computes the tensor rank, and determines the obstruction kernel for both the product and deformed metrics. Section4 presents the deformation family, derives the exact curvature formula with the factor $(\varepsilon^2 - 1)$, identifies the Bismut cancellation at the KK endpoint, and establishes the non-trivial lower bound for the physical metric through non-Bismut torsion representatives and the Riemannian holonomy. Section5 synthesizes these results into the main theorem and discusses its physical consequences, including the three-level irreducibility structure. Section7 provides illustrative examples, including the paradigmatic case of vanishing total Chern flux. Finally, Section9 summarizes the results and outlines directions for future work. *Technical details on the Levi-Civita connection and the explicit curvature computation are relegated to AppendicesA and B, respectively.*

2 Geometric Construction of the Extended Parameter Space

A generic two-component spin–orbit-coupled Bose–Einstein condensate (SOC BEC) confined in a quasi-two-dimensional toroidal trap is characterized by three independent, experimentally accessible degrees of freedom: the crystal momenta (k_x, k_y) spanning a Brillouin zone, a global $U(1)$ phase ϕ_+ associated with particle-number conservation, and a relative $U(1)$ phase ϕ_- between the two internal spin states [9, 11].

The natural mathematical framework for this system is a *principal bundle* with structure group $U(1)_+ \times U(1)_-$, whose total space M serves as the extended parameter space. Topologically, M is diffeomorphic to the four-dimensional torus T^4, but its geometric structure is that of a Kaluza–Klein manifold when equipped with the synthetic gauge fields.

2.1 Global geometric structure and the product approximation

For the purpose of applying the PT lower bound, we work in a framework where the parameter space admits a global product structure, possibly up to a smooth deformation. Although the parameter space $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ is a global topological product, the Berry curvatures $F^{(\pm)}$ are associated to the quantum eigenbundles over T^2_{BZ}, rather than to line bundles over M. Consequently, non-vanishing Chern classes $c_1^{(\pm)} \neq 0$ are fully compatible with the product structure of M.

Definition 2.1 (Extended parameter space as a product manifold)**.** *We consider the smooth manifold*

$$M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-} \cong T^4,$$

equipped with a family of Riemannian metrics $\{g_\varepsilon\}_{\varepsilon \in [0,1]}$ such that:

1. *For $\varepsilon = 0$, $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ is a Riemannian product metric.*
2. *For $\varepsilon = 1$, $g_1 = g_M$ is the physical Kaluza–Klein metric given by* (2.1)*.*
3. *The family is smooth in ε and preserves the orthogonal splitting $TM = \mathcal{H}_\varepsilon \oplus \mathcal{V}$, where $\mathcal{V} = \ker d\pi$ is the vertical distribution (independent of ε) and $\mathcal{H}_\varepsilon$ is the g_ε-orthogonal complement.*

Remark 2.2 (Topological vs. geometric product structure)**.** *While the underlying smooth manifold is a product, the physical metric g_M is not a Riemannian product unless the connections $A^{(\pm)}$ vanish. The family $\{g_\varepsilon\}$ interpolates between the geometric product structure ($\varepsilon = 0$) and the physical Kaluza–Klein structure ($\varepsilon = 1$). This deformation is essential to apply the PT theorem, which requires a genuine Riemannian product structure.*

2.2 Constant Berry curvature assumption

To make the geometric analysis fully explicit and to cover a wide range of experimentally relevant configurations, we work under the following assumption:

Assumption 2.3 (Constant Berry curvatures)**.** *The Berry curvature 2-forms are constant multiples of the Brillouin-zone volume form:*

$$F^{(\pm)} = dA^{(\pm)} = c^{(\pm)}\,\mathrm{vol}_{\mathrm{BZ}}, \qquad c^{(\pm)} \in \mathbb{R} \setminus \{0\},$$

where $\mathrm{vol}_{\mathrm{BZ}} = dk_x \wedge dk_y$. The constants $c^{(\pm)}$ are related to the Chern numbers by

$$c_1^{(\pm)} = \frac{1}{2\pi} \int_{T^2_{\mathrm{BZ}}} F^{(\pm)} = \frac{\mathrm{Area}(T^2_{\mathrm{BZ}})}{2\pi}\, c^{(\pm)}.$$

This condition is satisfied in numerous SOC BEC realizations, including linear Rashba–Dresselhaus couplings and uniform synthetic magnetic fields.

Remark 2.4 (Topological invariance)**.** *The cohomology classes $[F^{(\pm)}] \in H^2(T^2_{\mathrm{BZ}}; \mathbb{R})$ are topological invariants. Under Assumption2.3, these classes are proportional: $[F^{(-)}] = \lambda[F^{(+)}]$ with $\lambda = c^{(-)}/c^{(+)}$. Any deformation of the connection within its cohomology class preserves these invariants, ensuring the robustness of our results.*

2.3 The Kaluza–Klein metric and its deformation to a product

We construct a family of Riemannian metrics on M that interpolates between the physical Kaluza–Klein metric and a product metric. Let $g_{\mathrm{BZ}} = dk_x^2 + dk_y^2$ be the flat metric on the base torus T^2_{BZ}.

Definition 2.5 (Deformation family of Kaluza–Klein metrics)**.** *For $\varepsilon \in [0, 1]$, define the metric on M by:*

$$g_\varepsilon = \pi^* g_{\mathrm{BZ}} + \left(d\phi_+ + \varepsilon\, \pi^* A^{(+)}\right)^2 + \left(d\phi_- + \varepsilon\, \pi^* A^{(-)}\right)^2, \tag{2.1}$$

where $\pi : M \to T^2_{\mathrm{BZ}}$ is the projection onto the first factor. For $\varepsilon = 0$ we obtain the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, and for $\varepsilon = 1$ we obtain the physical Kaluza–Klein metric $g_1 = g_M$.

Remark 2.6 (Global well-definedness of the metric family)**.** *The metrics g_ε are* globally well-defined *on the total space M of the principal $U(1)_+ \times U(1)_-$-bundle over T^2_{BZ}. The 1-forms $A^{(\pm)}$ are connection 1-forms on local trivializations; the quantities $(d\phi_\pm + \varepsilon\pi^* A^{(\pm)})^2$ are* invariant under gauge transformations *$A^{(\pm)} \to A^{(\pm)} + d\lambda^{(\pm)}$ because the fiber coordinates $\phi_\pm$ transform as $\phi_\pm \to \phi_\pm - \varepsilon\lambda^{(\pm)}$. Consequently, the metric g_ε descends to a well-defined Riemannian metric on M even when the curvatures $F^{(\pm)}$ have non-zero Chern classes.*

Theorem 2.7 (Properties of the deformation family). *The family of metrics* $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ *defined on* M *satisfies the following properties:*

(i) Regularity: Each g_ε *is a smooth, positive-definite Riemannian metric on* M*. The projection* $\pi : (M, g_\varepsilon) \to (T^2_{\mathrm{BZ}}, g_{\mathrm{BZ}})$ *is a Riemannian submersion for every* $\varepsilon \in [0, 1]$*.*

(ii) Orthogonality: The vertical distribution $\mathcal{V} = \ker d\pi = \mathrm{span}\{\partial_{\phi_+}, \partial_{\phi_-}\}$ *is* g_ε*-orthogonal to the horizontal distribution* $\mathcal{H}_\varepsilon$*, where the latter is defined as the kernel of the connection 1-forms* $\Theta^{(\pm)}_\varepsilon = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$*.*

(iii) Fiber Isotropy: Each fibre $\pi^{-1}(k) \cong S^1 \times S^1$ *inherits a flat metric* $g_{\mathit{fibre}} = d\phi_+^2 + d\phi_-^2$ *which is independent of both the base point* $k \in T^2_{\mathrm{BZ}}$ *and the deformation parameter* ε*.*

(iv) Geodesic Geometry: The fibres are totally geodesic submanifolds of (M, g_ε) *for every* ε*, and the horizontal distribution* $\mathcal{H}_\varepsilon$ *has a curvature proportional to* $\varepsilon F^{(\pm)}$*.*

Proof. (i)-(ii) By construction, $g_\varepsilon = \pi^* g_{\mathrm{BZ}} + (\Theta^{(+)}_\varepsilon)^2 + (\Theta^{(-)}_\varepsilon)^2$. Since g_{BZ} is positive definite and the vertical forms are linearly independent, g_ε is a valid Riemannian metric. The orthogonality follows from the fact that $g_\varepsilon(X, V) = 0$ whenever $X \in \mathcal{H}_\varepsilon$ (so $\Theta^{(\pm)}_\varepsilon(X) = 0$) and $V \in \mathcal{V}$ (so $d\pi(V) = 0$).

(iii) Restricting g_ε to the vertical distribution $\mathcal{V}$ means evaluating it on vectors V such that $d\pi(V) = 0$. In this case, $\pi^* A^{(\pm)}(V) = A^{(\pm)}(d\pi(V)) = 0$, so $\Theta^{(\pm)}_\varepsilon|_{\mathcal{V}} = d\phi_\pm$. Thus $g_\varepsilon|_{\mathcal{V}} = d\phi_+^2 + d\phi_-^2$, which is constant and independent of ε.

(iv) The fibers are flat tori, and the connection potentials $A^{(\pm)}$ depend only on the base coordinates. The vanishing of the second fundamental form of the fibers follows from the Kaluza-Klein structure with constant fiber metrics. The curvature of the horizontal distribution is given by $d\Theta^{(\pm)}_\varepsilon = \varepsilon dA^{(\pm)} = \varepsilon F^{(\pm)}$, confirming that ε scales the synthetic magnetic field without altering the fiber geometry. □

2.4 Metric connection with totally skew-symmetric torsion

For each $\varepsilon \in [0, 1]$, we define a metric connection ∇^{C_ε} with torsion that encodes the physical Berry curvature.

Definition 2.8 (Torsion 3-form family). *For each* ε*, define the* torsion 3-form*:*

$$T_\varepsilon := F^{(+)} \wedge (d\phi_+ + \varepsilon\pi^* A^{(+)}) + F^{(-)} \wedge (d\phi_- + \varepsilon\pi^* A^{(-)}) \in \Omega^3(M).$$

Under Assumption2.3, this becomes:

$$T_\varepsilon = c^{(+)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_+ + \varepsilon\pi^* A^{(+)}) + c^{(-)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_- + \varepsilon\pi^* A^{(-)}).$$

Remark 2.9 (Pure bigrade structure of the torsion). *With respect to the product decomposition* $TM = V_1 \oplus V_2$*, where* $V_1 = T\,T^2_{\mathrm{BZ}}$ *and* $V_2 = T(S^1_{\varphi_+} \times S^1_{\varphi_-})$*, the torsion* T_ε *has* pure bigrade$(2,1)$*: every term belongs to* $\Gamma(\Lambda^2 V_1^* \otimes V_2^*)$*. Indeed, the constituents* $F^{(\pm)} \in \Gamma(\Lambda^2 V_1^*)$ *and* $d\varphi_\pm \in \Gamma(V_2^*)$*, while the additional terms* $F^{(\pm)} \wedge \varepsilon\pi^* A^{(\pm)}$ *would belong to* $\Gamma(\Lambda^3 V_1^*)$*, which vanishes identically since* $\dim V_1 = 2$*.*

Consequently, $T^{1,2}_\varepsilon \equiv 0$ *as a section for every* $\varepsilon \in [0, 1]$*, verifying the pure bigrade hypothesis of [12, Theorem 5.2]. This ensures that the overlap locus* $W^{2,1} \cap W^{1,2}$ *is empty and that Case 2 of the PT proof ([12, Theorem 5.2]) is vacuous for the SOCBEC model.*

Lemma 2.10 (Cohomological invariance of the torsion class)**.** *For all $\varepsilon \in [0,1]$, the de Rham cohomology class $[T_\varepsilon] \in H^3(M;\mathbb{R})$ is constant and equal to*

$$[T_\varepsilon] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-] \quad \textit{under the Künneth isomorphism.}$$

Proof. We verify that T_ε is closed for every $\varepsilon \in [0,1]$. Using $dF^{(\pm)} = 0$ (curvatures are closed) and $d(d\phi_\pm) = 0$, we compute:

$$dT_\varepsilon = -F^{(+)} \wedge d(\varepsilon\pi^* A^{(+)}) - F^{(-)} \wedge d(\varepsilon\pi^* A^{(-)}) = -\varepsilon \left(F^{(+)} \wedge F^{(+)} + F^{(-)} \wedge F^{(-)} \right).$$

Under Assumption2.3, $F^{(\pm)} = c^{(\pm)} dk_x \wedge dk_y$, hence

$$F^{(\pm)} \wedge F^{(\pm)} = c^{(\pm)2}(dk_x \wedge dk_y) \wedge (dk_x \wedge dk_y) = 0$$

by antisymmetry. Thus $dT_\varepsilon = 0$ for all ε. Under the Künneth decomposition

$$H^3(M;\mathbb{R}) \cong \left(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-}) \right) \oplus \left(H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\phi_+} \times S^1_{\phi_-}) \right),$$

the class $[T_0] = [F^{(+)} \wedge d\phi_+ + F^{(-)} \wedge d\phi_-]$ decomposes as

$$[T_0] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-]$$

in the $(2,1)$-component. The $(1,2)$-component $[T_0]_{1,2} \in H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\varphi_+} \times S^1_{\varphi_-})$ vanishes because every summand ofT_0 has the form $F^{(\pm)} \wedge d\varphi_\pm$, i.e. a 2-form on the base wedged with a 1-form on the fibre; no term of bidegree$(1,2)$ is present. For $T_\varepsilon = T_0 + \varepsilon(F^{(+)} \wedge \pi^* A^{(+)} + F^{(-)} \wedge \pi^* A^{(-)})$, the additional terms are 3-forms of bidegree $(3,0)$ in the Künneth decomposition, being wedge products of forms pulled back from T^2_{BZ}. Since

$$H^3(T^2_{\mathrm{BZ}}) = 0 \quad (\text{as } \dim T^2_{\mathrm{BZ}} = 2 < 3),$$

these terms contribute zero to the cohomology class. Therefore

$$[T_\varepsilon] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-] = [T_0]$$

for all $\varepsilon \in [0,1]$. □

Definition 2.11 (Metric connection family)**.** *For each $\varepsilon \in [0,1]$, define the $(2,1)$-tensor K_ε by:*

$$K_\varepsilon(X,Y,Z) := \frac{1}{2} T_\varepsilon(X,Y,Z), \quad X,Y,Z \in \mathfrak{X}(M).$$

The metric connection with totally skew-symmetric torsion *for parameter ε is:*

$$\nabla^{C_\varepsilon}_X Y := \nabla^{LC_\varepsilon}_X Y + K_\varepsilon(X,Y,\cdot)^{\sharp_\varepsilon},$$

where ∇^{LC_ε} is the Levi-Civita connection of g_ε and $\sharp_\varepsilon$ denotes the musical isomorphism $T^*M \to TM$ induced by g_ε.

Proposition 2.12 (Properties of the connection family)**.** *For each $\varepsilon \in [0,1]$, the connection ∇^{C_ε} satisfies:*

(i) $\nabla^{C_\varepsilon} g_\varepsilon = 0$ *(metric compatibility).*

(ii) The torsion tensor of ∇^{C_ε} *is exactly* T_ε*, i.e.:*

$$\nabla^{C_\varepsilon}_X Y - \nabla^{C_\varepsilon}_Y X - [X, Y] = T_\varepsilon(X, Y, \cdot)^{\sharp_\varepsilon}.$$

(iii) For $\varepsilon > 0$*,* $\nabla^{LC_\varepsilon} T_\varepsilon \neq 0$ *(non-parallel torsion), provided* $F^{(\pm)} \neq 0$*.*

Proof. Properties (i) and (ii) are standard consequences of the definition $\nabla^{C_\varepsilon} = \nabla^{LC_\varepsilon} + \frac{1}{2}T_\varepsilon$: adding a totally skew-symmetric (2, 1)-tensor to the Levi-Civita connection preserves metric compatibility, and the torsion of the resulting connection equals the skew-symmetric tensor (see, e.g., [15]).

For (iii), AppendixA shows that $\nabla^{LC_\varepsilon} d\phi_\pm = \frac{\varepsilon c^{(\pm)}}{2}(e^2 \otimes e^1 - e^1 \otimes e^2)$, which is nonzero for $\varepsilon > 0$ and $c^{(\pm)} \neq 0$. Since $T_\varepsilon = c^{(+)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_+ + \varepsilon\pi^* A^{(+)}) + c^{(-)} \operatorname{vol}_{\mathrm{BZ}} \wedge (d\phi_- + \varepsilon\pi^* A^{(-)})$ and the base form $\operatorname{vol}_{\mathrm{BZ}}$ is parallel, the Leibniz rule gives $\nabla^{LC_\varepsilon} T_\varepsilon \neq 0$. □

2.5 Parallel forms and the obstruction kernel

A crucial ingredient for the PT lower bound is the space of forms that are parallel with respect to the Levi-Civita connection of the product metric ($\varepsilon = 0$) and their behavior under deformation.

Definition 2.13 (Spaces of parallel forms on the factors)**.** *For the product metric* $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$*, we define:*

$$\mathcal{P}_2(T^2_{\mathrm{BZ}}) := \{\alpha \in \Omega^2(T^2_{\mathrm{BZ}}) \mid \nabla^{g_{\mathrm{BZ}}}\alpha = 0\} = \mathbb{R} \operatorname{vol}_{\mathrm{BZ}},$$
$$\mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}) := \{\beta \in \Omega^1(S^1_{\phi_+} \times S^1_{\phi_-}) \mid \nabla^{g_{\mathit{fibre}}}\beta = 0\} = \operatorname{span}\{d\phi_+, d\phi_-\}.$$

For $\varepsilon > 0$*, we define* $\mathcal{P}^\varepsilon_1(S^1_{\phi_+} \times S^1_{\phi_-})$ *as the space of 1-forms on* M *that are* g_ε*-parallel and vanish on* $\mathcal{H}_\varepsilon$ *(vertical parallel 1-forms).*

Theorem 2.14 (Deformation of vertical parallel 1-forms)**.** *Under Assumption2.3, for the deformation family* g_ε*, the space of parallel vertical 1-forms* $\mathcal{P}^\varepsilon_1(S^1_{\phi_+} \times S^1_{\phi_-})$ *satisfies:*

1. *For* $\varepsilon > 0$*, the space has dimension 1 and consists of constant-coefficient forms* $\eta = c_+ d\phi_+ + c_- d\phi_-$ *satisfying the algebraic constraint:*

$$c_+ + \lambda c_- = 0, \quad \textit{where } \lambda = \frac{c^{(-)}}{c^{(+)}}.$$

2. *For the product limit* $\varepsilon = 0$*, the constraint vanishes and the space has dimension 2, being spanned by* $\{d\phi_+, d\phi_-\}$*.*

Proof. The condition for a 1-form η to be parallel is $\nabla^{LC_\varepsilon}\eta = 0$. For a vertical form $\eta = c_+ d\phi_+ + c_- d\phi_-$, we evaluate the covariant derivative along horizontal directions. As derived in AppendixA using the connection 1-forms $\omega^a_{\ b}$ of the Kaluza–Klein metric g_ε, we have:

$$\nabla^{LC_\varepsilon}\eta = \frac{\varepsilon}{2}(c_+ c^{(+)} + c_- c^{(-)})(e^2 \otimes e^1 - e^1 \otimes e^2).$$

For any $\varepsilon > 0$, the vanishing of this expression requires $c_+c^{(+)}+c_-c^{(-)} = 0$, which is equivalent to $c_+ + \lambda c_- = 0$. This constraint defines a 1-dimensional subspace. At the point $\varepsilon = 0$, the entire expression vanishes regardless of the coefficients $c_\pm$, reflecting the fact that the Levi-Civita connection of the product metric decouples. This completes the proof (see AppendixA for the component-wise tensor derivation). □

Remark 2.15 (Global extension of vertical parallel forms)**.** *Since the vertical distribution* $\mathcal{V} = \ker d\pi$ *is* integrable *(being the kernel of a submersion) and the fibers* $\pi^{-1}(k)$ *are* totally geodesic *flat tori (Theorem2.7(iv)), any vertical 1-form* η *that is* g_ε*-parallel at one point* $p \in M$ *extends* uniquely *by parallel transport to a globally* g_ε*-parallel form. The algebraic constraint* $c_+ + \lambda c_- = 0$ *derived at one point in the proof of Theorem2.14 therefore holds* everywhere, *establishing that* $\dim \mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-}) = 1$ *globally for all* $\varepsilon > 0$*.*

Definition 2.16 (Obstruction kernel for the product metric)**.** *For the product metric* g_0*, let* $\mathcal{V}_{2,1}$ *be the span of the harmonic representative of the* $(2,1)$*-Künneth component of* $[T_0]$ *(this will be computed explicitly in Section3). Define:*

$$\mathcal{K}_0 := \left(\mathcal{V}_{2,1} \cap (\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}))\right) \oplus \left(\mathcal{V}_{1,2} \cap (\mathcal{P}_1(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_2(S^1_{\phi_+} \times S^1_{\phi_-}))\right).$$

Lemma 2.17 (Obstruction kernel for $\varepsilon > 0$)**.** *For* $\varepsilon > 0$*, let* $\mathcal{K}_\varepsilon$ *be the obstruction kernel defined using the space of parallel vertical 1-forms* $\mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$ *with respect to the metric* g_ε*. Under Assumption2.3, we have:*

$$\mathcal{K}_\varepsilon = 0 \quad \text{for all } \varepsilon > 0.$$

Proof. The $(2,1)$-component of the torsion class is represented by a simple tensor that will be explicitly computed in Section3; it has the form $\xi_0 = [\mathrm{vol}_{\mathrm{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])$ with $a_- = \lambda a_+$. The kernel $\mathcal{K}_\varepsilon$ is defined by the intersection of the span of its harmonic representative with the space $\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$.

As established in Theorem2.14, for $\varepsilon > 0$, any parallel vertical 1-form $\eta \in \mathcal{P}_1^\varepsilon$ must satisfy the algebraic constraint $c_+ + \lambda c_- = 0$, where $\lambda = c^{(-)}/c^{(+)}$. However, the harmonic representative has coefficients satisfying $a_- = \lambda a_+$. Substituting these into the parallel condition yields:

$$a_+ + \lambda a_- = a_+ + \lambda(\lambda a_+) = a_+(1 + \lambda^2).$$

Since $c^{(\pm)} \neq 0$, it follows that $1+\lambda^2 > 0$ (as $\lambda \in \mathbb{R}$). Thus, the parallel condition $a_+(1+\lambda^2) = 0$ is satisfied if and only if $a_+ = 0$, which implies $[\omega] = 0$. For any non-trivial torsion class, the intersection is therefore $\{0\}$, proving that $\mathcal{K}_\varepsilon = 0$. □

Remark 2.18 (Discontinuity of the obstruction kernel)**.** *The dimension of the obstruction kernel jumps at* $\varepsilon = 0$*: we have* $\dim \mathcal{K}_0 = 1$ *(as will be shown in Section3) while* $\dim \mathcal{K}_\varepsilon = 0$ *for all* $\varepsilon > 0$*. This discontinuity reflects the fact that the geometric product structure at* $\varepsilon = 0$ *allows the mixed class to be completely absorbed by parallel forms, while the Kaluza–Klein coupling for* $\varepsilon > 0$ *imposes additional holonomy constraints that prevent this absorption.*

2.6 Local adapted gauge

To perform explicit curvature calculations, we work in local gauges that simplify the metric at a given point.

Definition 2.19 (Adapted gauge at a point). *Let $p \in M$ with $\pi(p) = k_0 \in T^2_{\mathrm{BZ}}$. A gauge is called* adapted at p *if:*

(i) $A^{(\pm)}(k_0) = 0$,

(ii) $(d\phi_\pm + \varepsilon\pi^* A^{(\pm)})|_p = d\phi_\pm|_p$,

(iii) The metric g_ε satisfies $g_\varepsilon|_p = dk_x^2 + dk_y^2 + d\phi_+^2 + d\phi_-^2$.

Lemma 2.20 (Existence of adapted gauge for the family). *For any point $p \in M$ and for every value of the deformation parameter $\varepsilon \in [0,1]$, there exists a local gauge transformation such that the connection ∇^{C_ε} is adapted at p.*

Proof. By Definition2.5, the connection 1-forms of the family are given by $\Theta^{(\pm)}_\varepsilon = d\phi_\pm + \varepsilon\pi^* A^{(\pm)}$. An adapted gauge at p requires that the connection 1-forms satisfy $\Theta^{(\pm)}_\varepsilon|_p = d\phi_\pm|_p$, which is equivalent to the vanishing of the local potentials $\varepsilon A^{(\pm)}$ at the point p.

For any $\varepsilon \in [0,1]$, consider the standard gauge transformation $A^{(\pm)} \to A^{(\pm)} + df^{(\pm)}$. By choosing the functions $f^{(\pm)}$ such that $df^{(\pm)}|_p = -A^{(\pm)}|_p$ (which is always possible in a local neighborhood of p), we obtain a transformed potential that vanishes at p. Since the parameter ε scales the potential linearly, the condition $\varepsilon(A^{(\pm)} + df^{(\pm)})|_p = 0$ is satisfied for all ε. Thus, a single gauge transformation, independent of ε, suffices to make the entire family adapted at p. □

2.7 Summary of the geometric framework

We have established the following geometric framework:

1. The extended parameter space is the smooth product manifold $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, equipped with a one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ interpolating between a Riemannian product metric ($\varepsilon = 0$) and the physical Kaluza–Klein metric ($\varepsilon = 1$).

2. For each ε, we have a metric connection ∇^{C_ε} with totally skew-symmetric torsion T_ε. The cohomology class $[T_\varepsilon]$ is constant in ε and admits a Künneth decomposition into mixed components (Lemma2.10).

3. The space of vertical parallel 1-forms undergoes a discontinuous jump at $\varepsilon = 0$: $\dim \mathcal{P}^0_1 = 2$ while $\dim \mathcal{P}^\varepsilon_1 = 1$ for all $\varepsilon > 0$ (Theorem2.14 and Lemma2.15).

4. Consequently, the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes for all $\varepsilon > 0$ (Lemma2.17), while $\mathcal{K}_0$ is non-trivial. This discontinuity is the key to obtaining a non-trivial lower bound for the physical connection.

In the next section, we compute the topological invariants (mixed cohomology class and its tensor rank) explicitly and determine the reduced rank $r^\sharp_\varepsilon = \mathrm{rank}_{\mathbb{R}}([T_\varepsilon]_{\mathrm{mixed}}) - \dim \mathcal{K}_\varepsilon$ for both $\varepsilon = 0$ and $\varepsilon > 0$.

3 Cohomology of the Extended Space and the Mixed Tensor Rank

In this section, we analyze the cohomological structure of the extended parameter space M and compute the topological invariants that enter the PT lower bound [12]. Crucially, we distinguish between:

- *Topological invariants* (independent of the metric): the cohomology class $[\omega]$ and its mixed tensor rank.
- *Geometric data* (depending on the metric): the spaces of parallel forms $\mathcal{P}_k$ and the obstruction kernel $\mathcal{K}$.

Throughout, we work under Assumption2.3 (constant Berry curvatures).

3.1 De Rham cohomology and Künneth decomposition

Since $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ is a product manifold (as a smooth manifold), its de Rham cohomology is given by the Künneth theorem:

Theorem 3.1 (Cohomology of M)**.** *The de Rham cohomology groups of M are:*

$$\begin{aligned} H^0(M;\mathbb{R}) &\cong \mathbb{R}, \\ H^1(M;\mathbb{R}) &\cong \mathbb{R}^4, \\ H^2(M;\mathbb{R}) &\cong \mathbb{R}^6, \\ H^3(M;\mathbb{R}) &\cong \mathbb{R}^4, \\ H^4(M;\mathbb{R}) &\cong \mathbb{R}. \end{aligned}$$

Moreover, the Künneth isomorphism gives a canonical decomposition:

$$H^3(M;\mathbb{R}) \cong \bigoplus_{p+q=3} H^p(T^2_{\mathrm{BZ}};\mathbb{R}) \otimes H^q(S^1_{\phi_+} \times S^1_{\phi_-};\mathbb{R}).$$

Explicitly, the non-zero summands are:

$$H^3(M;\mathbb{R}) \cong \left(H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-})\right) \oplus \left(H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\phi_+} \times S^1_{\phi_-})\right).$$

The torsion 3-form T defined in Definition2.8 (for $\varepsilon = 1$) represents a well-defined cohomology class:

Definition 3.2 (Torsion cohomology class)**.** *The torsion cohomology class is:*

$$[\omega] := [T] \in H^3(M;\mathbb{R}).$$

Under the deformation family g_ε, the class $[\omega]$ is constant in ε, as shown in Lemma2.10.

3.2 Harmonic representative for the product metric

For the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, the harmonic representative of $[\omega]$ can be written explicitly.

Theorem 3.3 (Harmonic representative for g_0)**.** *Under Assumption2.3, the unique g_0-harmonic representative of $[\omega]$ is:*

$$\omega_h^0 = \mathrm{vol}_{\mathrm{BZ}} \wedge (a_+ d\phi_+ + a_- d\phi_-),$$

where the constants $a_\pm \in \mathbb{R}$ satisfy:

$$a_- = \lambda a_+, \quad \lambda = \frac{c^{(-)}}{c^{(+)}}.$$

These constants are uniquely determined by the cohomology classes $[F^{(\pm)}]$.

Proof. Since g_0 is the flat product metric on T^4, the space of harmonic k-forms is exactly the space of constant-coefficient k-forms (see, e.g., [15]). The torsion 3-form at $\varepsilon = 0$ is

$$T_0 \;=\; c^{(+)}\,\mathrm{vol}_{\mathrm{BZ}} \wedge d\varphi_+ \;+\; c^{(-)}\,\mathrm{vol}_{\mathrm{BZ}} \wedge d\varphi_-\,,$$

which already has constant coefficients with respect to the standard coframe $\{dk_x, dk_y, d\varphi_+, d\varphi_-\}$ of (T^4, g_0). Hence T_0 is itself the g_0-harmonic representative:

$$\omega_h^0 \;=\; \mathrm{vol}_{\mathrm{BZ}} \wedge \big(a_+\, d\varphi_+ + a_-\, d\varphi_-\big), \qquad a_+ := c^{(+)}, \quad a_- := c^{(-)}.$$

The ratio $a_-/a_+ = c^{(-)}/c^{(+)} =: \lambda$ is determined by the cohomology classes $[F^{(\pm)}]$. Uniqueness follows from the Hodge theorem applied to the flat torus. □

Remark 3.4 (Harmonic representative for $\varepsilon > 0$)**.** *For $\varepsilon > 0$, the harmonic representative ω_h^ε with respect to g_ε is not a wedge product of forms on the factors, because the metric is not a product. However, its cohomology class is the same as ω_h^0.*

3.3 Mixed tensor rank

We now define the mixed tensor rank for a cohomology class in $H^3(M)$. This is a purely topological invariant.

Definition 3.5 (Mixed tensor rank)**.** *Let $[\eta] \in H^3(M; \mathbb{R})$. Under the Künneth decomposition, write*

$$[\eta] = \sum_{i=1}^{r_{2,1}} [\alpha_i] \otimes [\beta_i] + \sum_{j=1}^{r_{1,2}} [\tilde{\alpha}_j] \otimes [\tilde{\beta}_j],$$

where $[\alpha_i] \in H^2(T^2_{\mathrm{BZ}})$, $[\beta_i] \in H^1(S^1_{\phi_+} \times S^1_{\phi_-})$, and similarly for the $(1,2)$-part. The mixed tensor rank *of $[\eta]$ is*

$$\mathrm{rank}_{\mathbb{R}}([\eta]_{\mathrm{mixed}}) := \min\{r_{2,1} + r_{1,2}\},$$

where the minimum is taken over all such representations.

Theorem 3.6 (Mixed tensor rank of the torsion class)**.** *Under Assumption2.3, the mixed tensor rank of $[\omega]$ is*

$$\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1.$$

Proof. From the explicit form of the harmonic representative ω_h^0 (Theorem3.3), we have

$$[\omega] = [\mathrm{vol}_{\mathrm{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])\,.$$

Since $a_+[d\phi_+] + a_-[d\phi_-]$ is a single element in $H^1(S^1_{\phi_+} \times S^1_{\phi_-})$, this shows that the mixed rank is at most 1. It cannot be 0 because $[\omega] \neq 0$ (under Assumption2.3, $c^{(\pm)} \neq 0$). □

Remark 3.7 (Geometric interpretation)**.** *The rank being 1 indicates that the topological coupling between the Brillouin zone and the phase directions is irreducible: the torsion class cannot be decomposed into two independent tensor products. This reflects the fact that the two U(1) sectors are not independent topologically when their curvatures are proportional.*

3.4 Parallel forms and obstruction kernel for the product metric

To apply the PT theorem, we need the spaces of parallel forms for the product metric g_0.

Lemma 3.8 (Parallel forms on the factors for g_0)**.** *For the product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$, we have:*

$$\mathcal{P}_2(T^2_{\mathrm{BZ}}, g_{\mathrm{BZ}}) = \mathbb{R}\ \mathrm{vol}_{\mathrm{BZ}},$$
$$\mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}, g_0|_{\mathit{fibre}}) = \mathrm{span}\{d\phi_+, d\phi_-\}.$$

Proof. The base is flat, so all harmonic forms are parallel. The fibre is a flat torus, so all constant 1-forms are parallel. □

Now we compute the obstruction kernel $\mathcal{K}_0$ for the product metric.

Definition 3.9 (Obstruction kernel for g_0)**.** *Let $\mathcal{V}_{2,1}$ be the subspace of $H^2(T^2_{\mathrm{BZ}}) \otimes H^1(S^1_{\phi_+} \times S^1_{\phi_-})$ spanned by the harmonic representatives of the $(2,1)$-component of $[\omega]$ with respect to g_0 (as given in Definition3.5). Define*

$$\mathcal{K}_0 := \Big(\mathcal{V}_{2,1} \cap (\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}))\Big) \oplus \Big(\mathcal{V}_{1,2} \cap (\mathcal{P}_1(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_2(S^1_{\phi_+} \times S^1_{\phi_-}))\Big)\,.$$

Theorem 3.10 (Obstruction kernel for g_0)**.** *Under Assumption2.3, for the product metric g_0, we have*

$$\dim \mathcal{K}_0 = 1.$$

Proof. From Theorem3.3, the harmonic representative of the $(2,1)$-component is

$$\xi_0 = \mathrm{vol}_{\mathrm{BZ}} \otimes (a_+ d\phi_+ + a_- d\phi_-).$$

Thus $\mathcal{V}_{2,1} = \mathrm{span}\{\xi_0\}$. Since $\mathrm{vol}_{\mathrm{BZ}} \in \mathcal{P}_2(T^2_{\mathrm{BZ}})$ and $a_+ d\phi_+ + a_- d\phi_- \in \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-})$ (it is a constant linear combination), we have

$$\xi_0 \in \mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}).$$

Hence

$$\mathcal{V}_{2,1} \subseteq \mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-}),$$

so the intersection is all of $\mathcal{V}_{2,1}$, and $\dim \mathcal{K}_0 = \dim \mathcal{V}_{2,1} = 1$. The $(1,2)$-component $[\omega]_{1,2} \in H^1(T^2_{\mathrm{BZ}}) \otimes H^2(S^1_{\varphi_+} \times S^1_{\varphi_-})$ vanishes because the torsion has pure bigrade$(2,1)$ (Remark2.9), so it contributes nothing to$\mathcal{K}_0$. □

3.5 Reduced rank for the product metric

We now compute the key quantity that appears in the PT lower bound for the product metric.

Definition 3.11 (Reduced rank for g_0). *The reduced rank for the product metric g_0 is*

$$r_0^\sharp := \operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) - \dim \mathcal{K}_0.$$

Corollary 3.12 (Value of $r_0^\sharp$). *Under Assumption2.3,*

$$r_0^\sharp = 0.$$

Proof. By Theorem3.6, $\operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) = 1$. By Theorem3.10, $\dim \mathcal{K}_0 = 1$. Hence $r_0^\sharp = 1 - 1 = 0$. ☐

Remark 3.13 (Interpretation). *For the product metric g_0, the PT theorem gives the lower bound*

$$\dim \mathfrak{hol}^{\text{off}}(\nabla^{C_0}) \geq r_0^\sharp = 0,$$

which is trivial. This is expected because for $\varepsilon = 0$, the connection ∇^{C_0} is adapted to the product structure and its curvature may be block-diagonal. The non-trivial bound will arise from the deformation to $\varepsilon > 0$.

3.6 Behavior under deformation

We now examine how the obstruction kernel changes as we move away from the product metric.

Theorem 3.14 (Obstruction kernel for $\varepsilon > 0$). *For $\varepsilon > 0$, under Assumption2.3, the obstruction kernel $\mathcal{K}_\varepsilon$ (defined analogously using the spaces of parallel forms for g_ε) satisfies*

$$\mathcal{K}_\varepsilon = 0.$$

Consequently, $\dim \mathcal{K}_\varepsilon = 0$.

Proof. The $(2,1)$-component of the torsion class is represented by the simple tensor $\xi_0 = [\text{vol}_{\text{BZ}}] \otimes (a_+[d\phi_+] + a_-[d\phi_-])$ (Theorem3.3). The kernel $\mathcal{K}_\varepsilon$ is defined by the intersection of the span of its harmonic representative with the space $\mathcal{P}_2(T^2_{\text{BZ}}) \otimes \mathcal{P}_1^\varepsilon(S^1_{\phi_+} \times S^1_{\phi_-})$.

As established in Theorem2.14, for $\varepsilon > 0$, any parallel vertical 1-form $\eta \in \mathcal{P}_1^\varepsilon$ must satisfy the algebraic constraint $c_+ + \lambda c_- = 0$, where $\lambda = c^{(-)}/c^{(+)}$. However, the harmonic representative of $[\omega]$ has coefficients satisfying $a_- = \lambda a_+$. Substituting these into the parallel condition yields:

$$a_+ + \lambda a_- = a_+ + \lambda(\lambda a_+) = a_+(1 + \lambda^2).$$

Since $c^{(\pm)} \neq 0$ (Assumption2.3), it follows that $1 + \lambda^2 > 0$ for any real λ. Thus, the parallel condition $a_+(1 + \lambda^2) = 0$ is satisfied if and only if $a_+ = 0$, which implies $[\omega] = 0$. For any non-trivial torsion class, the intersection is therefore $\{0\}$, proving that $\mathcal{K}_\varepsilon = 0$. ☐

Corollary 3.15 (Reduced rank for $\varepsilon > 0$). *For $\varepsilon > 0$, the reduced rank $r_\varepsilon^\sharp := \operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) - \dim \mathcal{K}_\varepsilon$ satisfies*

$$r_\varepsilon^\sharp = 1.$$

Proof. By Theorem3.6, $\operatorname{rank}_{\mathbb{R}}([\omega]_{\text{mixed}}) = 1$. By Theorem3.14, $\dim \mathcal{K}_\varepsilon = 0$ for $\varepsilon > 0$. Hence $r_\varepsilon^\sharp = 1 - 0 = 1$. ☐

3.7 Summary

We have established the following topological and geometric data:

1. The torsion class $[\omega] \in H^3(M;\mathbb{R})$ has mixed tensor rank $\mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1$ (Theorem3.6), a topological invariant.

2. For the product metric g_0, the obstruction kernel $\mathcal{K}_0$ has dimension 1, giving reduced rank $r_0^\sharp = 0$ (Corollary3.12).

3. For the deformed metrics g_ε with $\varepsilon > 0$, the obstruction kernel vanishes, giving reduced rank $r_\varepsilon^\sharp = 1$ (Corollary3.15).

4. The PT theorem applies directly to (M, g_0, ∇^{C_0}), yielding

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \geq r_0^\sharp = 0.$$

   This bound is trivial, but for $\varepsilon > 0$ the bound becomes non-trivial, as will be established in Section4 through direct curvature analysis.

In the next section, we apply the deformation strategy and establish the non-trivial lower bound for the physical connection $\nabla^C = \nabla^{C_1}$.

4 Applying the PT Theorem via Deformation and Direct Analysis

The PT lower bound [12] applies directly to genuine Riemannian product manifolds. Our physical metric g_M is a Kaluza–Klein metric that is not a product. However, we have constructed a smooth deformation g_ε (Definition2.5) that interpolates between the product metric g_0 and the physical metric $g_M = g_1$. In this section, we:

1. Apply the original PT theorem to the product metric g_0 at $\varepsilon = 0$.

2. Establish the lower bound for $\varepsilon \in (0, 1)$ through an exact curvature computation in adapted gauge, exhibiting the closed-form dependence on ε.

3. Analyse the Kaluza–Klein endpoint $\varepsilon = 1$, where a structurally distinct phenomenon—the Bismut cancellation—must be taken into account, and establish the lower bound for the physical metric through a non-Bismut torsion representative.

The key observation is that while the PT bound is trivial at the product limit ($\varepsilon = 0$), it becomes non-trivial for all $\varepsilon > 0$ due to the vanishing of the obstruction kernel (Lemma2.17). The exact curvature formula derived in Theorem4.5 reveals that the off-diagonal curvature of the natural Kaluza–Klein torsion is proportional to the factor $(\varepsilon^2 - 1)$, which vanishes at the single point $\varepsilon = 1$. This vanishing is identified as the Bismut cancellation of the Kaluza–Klein structure and is shown to be non-generic among torsion representatives of the same cohomology class.

4.1 Applying the PT theorem to the product metric g_0

For $\varepsilon = 0$, the metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ is a Riemannian product. The corresponding connection ∇^{C_0} has torsion $T_0 = F^{(+)} \wedge d\phi_+ + F^{(-)} \wedge d\phi_-$. The PT theorem can be applied directly to this geometric setup.

Theorem 4.1 (PT bound for the product metric)**.** *For the product metric g_0 and the connection ∇^{C_0}, the PT theorem gives*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \;\geq\; r_0^\sharp \;=\; 0.$$

Proof. The PT theorem (Theorem 5.2 of [12]) states that

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \;\geq\; \mathrm{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) - \dim \mathcal{K}_0.$$

From Corollary3.12 (Section3) we have

$$r_0^\sharp = \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}_0 = 1 - 1 = 0.$$

□

Remark 4.2 (Triviality at the product limit)**.** *The bound is trivial for the product metric, as expected. In the product geometry, the topological obstruction encoded by the mixed class is completely absorbed by the parallel forms: the harmonic representative of $[\omega]$ belongs to $\mathcal{P}_2(T^2_{\mathrm{BZ}}) \otimes \mathcal{P}_1(S^1_{\phi_+} \times S^1_{\phi_-})$, yielding $\dim \mathcal{K}_0 = 1$ and hence $r_0^\sharp = 0$. This absorption is the geometric reason for the vanishing of the lower bound at $\varepsilon = 0$. We emphasise that the PT bound is a* lower *bound: as the exact computation below will show, the connection ∇^{C_0} does possess non-trivial off-diagonal curvature (generated by the quadratic torsion term), even though the cohomological bound does not detect it.*

4.2 Exact curvature analysis for the deformation family

For $\varepsilon > 0$, the metric g_ε is no longer a Riemannian product, and the PT theorem does not apply directly. However, we can establish the lower bound through explicit pointwise computation of the off-diagonal curvature components for the SOC BEC model.

The key geometric fact is that for $\varepsilon > 0$, the space of vertical parallel 1-forms drops from dimension 2 to dimension 1 (Theorem2.14), causing the obstruction kernel to vanish: $\mathcal{K}_\varepsilon = 0$ (Lemma2.17). This yields a non-trivial reduced rank $r_\varepsilon^\sharp = 1$ for all $\varepsilon > 0$.

Definition 4.3 (Off-diagonal curvature test for $\varepsilon > 0$)**.** *Fix $\varepsilon > 0$ and a point $p \in M$. Let $\mathcal{H}_p$ and $\mathcal{V}_p$ denote the horizontal and vertical subspaces of T_pM, respectively, and let*

$$\pi_{\mathrm{off}} : \mathfrak{so}(T_pM) \;\longrightarrow\; \mathrm{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \mathrm{Hom}(\mathcal{V}_p, \mathcal{H}_p)$$

be the projection onto the off-diagonal component (endomorphisms that map $\mathcal{H}_p \to \mathcal{V}_p$ and $\mathcal{V}_p \to \mathcal{H}_p$).

Define the off-diagonal curvature test *at p as the collection of endomorphisms*

$$\mathcal{R}_p^{\mathrm{off}}(\varepsilon) \;:=\; \big\{\, \pi_{\mathrm{off}}\big(R^{C_\varepsilon}(X, Z)\big) \;\big|\; X \in \mathcal{H}_p,\; Z \in \mathcal{V}_p \,\big\} \;\subset\; \mathrm{Hom}(\mathcal{H}_p, \mathcal{V}_p) \oplus \mathrm{Hom}(\mathcal{V}_p, \mathcal{H}_p).$$

The off-diagonal curvature is non-trivial at p *if $\mathcal{R}_p^{\mathrm{off}}(\varepsilon) \neq \{0\}$, i.e. if at least one mixed-input curvature operator has a nonzero off-diagonal projection.*

Remark 4.4 (Intrinsic nature of the off-diagonal curvature test). *Although explicit computations are performed in an adapted gauge (Definition2.19), the set $\mathcal{R}_p^{\mathrm{off}}(\varepsilon)$ is intrinsically defined because: the curvature tensor $R^{C_\varepsilon}(X,Z)$ is a $(1,3)$-tensor independent of any choice of coordinates or gauge; the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$ is determined by the Kaluza–Klein structure via $\mathcal{V}_p = \ker d\pi|_p$ and $\mathcal{H}_p = \mathcal{V}_p^{\perp_{g_\varepsilon}}$; and the projection π_{off} depends only on the splitting, not on the choice of basis within each factor. By the Ambrose–Singer theorem, if $\mathcal{R}_p^{\mathrm{off}}(\varepsilon) \neq \{0\}$, then $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \neq \{0\}$ and hence $\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$.*

Theorem 4.5 (Exact off-diagonal curvature for the deformation family). *Consider the extended parameter space $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ equipped with the Kaluza–Klein metric g_ε and the metric connection $\nabla^{C_\varepsilon} = \nabla^{\mathrm{LC}(\varepsilon)} + \frac{1}{2}T_\varepsilon$ with the natural Kaluza–Klein torsion of Definition2.8.*

Under Assumption2.3, the curvature of ∇^{C_ε} evaluated on the mixed pair (e_1, e_3) satisfies the exact identity

$$R^{C_\varepsilon}(e_1,e_3)\, e_3 \;=\; \frac{(c^{(+)})^2\,(\varepsilon^2-1)}{4}\, e_1, \tag{4.1}$$

valid for all $\varepsilon \in [0,1]$. More generally, the full off-diagonal curvature endomorphism $R^{C_\varepsilon}(e_1,e_3)$ is given by

$$R^{C_\varepsilon}(e_1,e_3) \;:\; \begin{cases} e_1 \;\mapsto\; -\dfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_3 \;-\; \dfrac{c^{(+)}c^{(-)}(\varepsilon^2-1)}{4}\, e_4, \\ e_3 \;\mapsto\; \dfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_1, \\ e_4 \;\mapsto\; \dfrac{c^{(+)}c^{(-)}(\varepsilon^2-1)}{4}\, e_1, \end{cases} \tag{4.2}$$

with $R^{C_\varepsilon}(e_1,e_3)\, e_2 = 0$. Every nonzero entry is purely off-diagonal, and all components share the common factor $(\varepsilon^2 - 1)$.

Consequently, for all $\varepsilon \in [0,1)$ and all $c^{(+)} \neq 0$,

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \;\geq\; 1 \qquad \text{at every point } p \in M. \tag{4.3}$$

Proof. We evaluate the three terms in the curvature decomposition

$$R^{C_\varepsilon}(X,Y) \;=\; R^{\mathrm{LC}(\varepsilon)}(X,Y) \;+\; \tfrac{1}{2}\big[(\nabla_X^{\mathrm{LC}(\varepsilon)}T_\varepsilon)(Y,\cdot) - (\nabla_Y^{\mathrm{LC}(\varepsilon)}T_\varepsilon)(X,\cdot)\big]^\sharp \;+\; \tfrac{1}{4}[T_X, T_Y]$$

for the mixed inputs $X = e_1 \in \mathcal{H}_p$, $Y = e_3 \in \mathcal{V}_p$, using the corrected Levi-Civita connection derived in AppendixA.

1. Levi-Civita connection with the horizontal–horizontal form. The complete set of connection 1-forms for the Kaluza–Klein metric g_ε includes, in addition to the vertical–horizontal forms ${\omega^3}_a$, ${\omega^4}_a$, the horizontal–horizontal form

$${\omega^1}_2 \;=\; -\frac{\varepsilon\, c^{(+)}}{2}\, e^3 \;-\; \frac{\varepsilon\, c^{(-)}}{2}\, e^4, \tag{4.4}$$

which is non-zero for $\varepsilon > 0$ (see AppendixA for the derivation from the first structure equation). This form is responsible for the non-vanishing of the Levi-Civita curvature on mixed inputs and was absent in early versions of this computation; its inclusion is essential for obtaining the exact formula.

2. Term I: Levi-Civita curvature. Since ${\omega^1}_2$ is proportional to e^3 and e^4, it evaluates non-trivially on vertical vectors. The curvature 2-forms that contribute to $R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)$ are computed in AppendixB; the result is

$$R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)\, e_3 \;=\; \frac{\varepsilon^2\, (c^{(+)})^2}{4}\, e_1. \tag{4.5}$$

This is purely off-diagonal ($\mathcal{V} \to \mathcal{H}$) and of order $\mathcal{O}(\varepsilon^2)$.

3. Term II: torsion-derivative contribution. The torsion $T_\varepsilon|_p = c^{(+)} e^1 \wedge e^2 \wedge e^3 + c^{(-)} e^1 \wedge e^2 \wedge e^4$ has constant coefficients in the orthonormal frame at p in adapted gauge. However, the covariant derivative $\nabla^{\mathrm{LC}(\varepsilon)} T_\varepsilon$ is not identically zero because the connection 1-form ${\omega^1}_2$ introduces Christoffel corrections when differentiating in the vertical direction. Evaluating on the triple $(e_1, e_3; e_3)$, the off-diagonal projection of the torsion-derivative term vanishes identically for every ε:

$$\big\langle e_1,\ \tfrac{1}{2}\big[(\nabla^{\mathrm{LC}(\varepsilon)}_{e_1} T_\varepsilon)(e_3, e_3, \cdot) - (\nabla^{\mathrm{LC}(\varepsilon)}_{e_3} T_\varepsilon)(e_1, e_3, \cdot)\big]^\sharp \big\rangle = 0. \tag{4.6}$$

This follows from the antisymmetry $T_\varepsilon(e_3, e_3, \cdot) = 0$ in the first summand and from the explicit Christoffel corrections in the second (see AppendixB for the detailed computation).

4. Term III: quadratic torsion. The commutator $[T_{e_1}, T_{e_3}]$ is computed entry-by-entry in PropositionB.6 of AppendixB. Its contribution to the off-diagonal curvature is

$$\tfrac{1}{4}[T_{e_1}, T_{e_3}]\, e_3 \;=\; -\frac{(c^{(+)})^2}{4}\, e_1, \tag{4.7}$$

which is independent of ε and purely off-diagonal.

5. Combining the three terms. Adding (4.5), (4.6), and (4.7):

$$R^{C_\varepsilon}(e_1, e_3)\, e_3 \;=\; \frac{\varepsilon^2\, (c^{(+)})^2}{4}\, e_1 \;+\; 0 \;-\; \frac{(c^{(+)})^2}{4}\, e_1 \;=\; \frac{(c^{(+)})^2\, (\varepsilon^2 - 1)}{4}\, e_1. \tag{4.8}$$

This establishes (4.1). The remaining entries (4.2) are obtained by an analogous computation on all basis vectors (AppendixB); each is proportional to $(\varepsilon^2 - 1)$.

By homogeneity of the model (PropositionB.2), the curvature components are constant on M. For $\varepsilon \in [0, 1)$ and $c^{(+)} \neq 0$, the factor $(\varepsilon^2 - 1) \neq 0$, so $\pi_{\mathrm{off}}(R^{C_\varepsilon}(e_1, e_3)) \neq 0$. By the Ambrose–Singer theorem, this operator belongs to $\mathfrak{hol}_p(\nabla^{C_\varepsilon})$, and its off-diagonal projection is non-zero, yielding the bound (4.3). □

Corollary 4.6 (Lower bound for $\varepsilon \in (0, 1)$)**.** *For every $\varepsilon \in (0, 1)$ and every point $p \in M$,*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \;\geq\; 1.$$

Proof. For $\varepsilon \in (0, 1)$, the factor $(\varepsilon^2 - 1) < 0$ is non-zero. The bound follows from Theorem4.5. □

The exact formula (4.1) reveals a fundamental structural feature at the Kaluza–Klein endpoint $\varepsilon = 1$: the factor $(\varepsilon^2 - 1)$ vanishes, and the entire off-diagonal curvature of the natural KK torsion is identically zero. The next subsection analyses this phenomenon and shows that it is non-generic.

4.3 The Bismut cancellation and non-Bismut torsion representatives

At $\varepsilon = 1$, the exact formula (4.1) gives $R^{C_1}(e_1, e_3) = 0$ for the natural Kaluza–Klein torsion $T_1 = F^{(+)} \wedge \Theta^{(+)} + F^{(-)} \wedge \Theta^{(-)}$. We now identify this vanishing as the Bismut cancellation, show it is confined to a single torsion representative, and establish the lower bound for the physical metric through an alternative representative.

The cancellation has a transparent origin: the Levi-Civita connection of the KK metric satisfies $\nabla^{\mathrm{LC}(1)}_{e_i} e_\alpha = \frac{c}{2} e_\perp$ for horizontal e_i and vertical e_α, while the contorsion contributes $\frac{1}{2} T_{e_i}(e_\alpha) = -\frac{c}{2} e_\perp$. At $\varepsilon = 1$ these terms cancel exactly:

$$\nabla^{C_1}_{e_i} e_\alpha \;=\; \nabla^{\mathrm{LC}(1)}_{e_i} e_\alpha + \tfrac{1}{2} T_{e_i}(e_\alpha) \;=\; 0 \qquad \text{for all } i \in \{1,2\},\ \alpha \in \{1,2,3,4\}. \tag{4.9}$$

Parallel transport along horizontal paths therefore preserves the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$, and the holonomy of ∇^{C_1} is reducible: $\mathfrak{hol}^{\mathrm{off}}_p(\nabla^{C_1}) = \{0\}$. This is a well-known feature of Kaluza–Klein geometry: the natural torsion of a principal bundle is the *Bismut torsion*, the unique totally skew-symmetric torsion for which the horizontal distribution is parallel along horizontal directions. We denote this torsion by T_{Bis}.

The Bismut cancellation is *not* a property of the cohomology class $[\omega]$ but of the specific representative T_{Bis}. We now show that it is non-generic: every other representative of $[\omega]$ produces non-trivial off-diagonal holonomy.

Theorem 4.7 (Lower bound for the physical metric with non-Bismut torsion)**.** *Let (M, g_1) be the extended parameter space equipped with the physical Kaluza–Klein metric, and let $\nabla^{C'} = \nabla^{\mathrm{LC}(1)} + \frac{1}{2} T'$ be any cohomologically calibrated metric connection with totally skew-symmetric torsion satisfying $[T'] = [\omega]$ and $T' \neq T_{\mathrm{Bis}}$. Under Assumption2.3, denote by $\psi : M \to \mathbb{R}$ the function defined by*

$$T' \;=\; \big(c^{(+)} + \psi\big)\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_+ \;+\; c^{(-)}\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_- \;+\; \eta, \tag{4.10}$$

where $\eta \in \Gamma(\Lambda^3 T^*M)$ collects terms of bidegree $(3,0)$ and $(1,2)$, and ψ captures the deviation of the $(2,1)$-component from T_{Bis}. Then:

(i) *On the non-empty open set $U_\psi := \{p \in M : \psi(p) \neq 0\}$, the off-diagonal curvature satisfies the exact identity*

$$R^{C'}(e_1, e_3)\, e_3\big|_p \;=\; -\frac{\psi(p)\big(2c^{(+)} + \psi(p)\big)}{4}\, e_1 \;\neq\; 0, \tag{4.11}$$

and consequently $\dim \mathfrak{hol}^{\mathrm{off}}_p(\nabla^{C'}) \geq 1$ for every $p \in U_\psi$.

(ii) *The Bismut torsion T_{Bis} (corresponding to $\psi \equiv 0$) is the unique representative of $[\omega]$ of pure bidegree $(2,1)$ for which $\mathfrak{hol}^{\mathrm{off}}_p(\nabla^{C}) = \{0\}$ at every point.*

(iii) *Even for the Bismut connection ∇^{C_1}, the Levi-Civita curvature of g_1 retains a non-trivial off-diagonal component:*

$$R^{\mathrm{LC}(1)}(e_1, e_3)\, e_3 \;=\; \frac{(c^{(+)})^2}{4}\, e_1 \;\neq\; 0, \tag{4.12}$$

certifying that the Riemannian holonomy of (M, g_1) is irreducible with respect to the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$.

Proof. Proof of (i). The Levi-Civita connection $\nabla^{\mathrm{LC}(1)}$ depends only on the metric g_1, not on the torsion, and is unchanged. The modified contorsion is $\frac{1}{2}T'_{e_1}(e_3) = -\frac{1}{2}(c^{(+)} + \psi)\, e_2$. At $\varepsilon = 1$, the LC contribution is $\nabla^{\mathrm{LC}(1)}_{e_1} e_3 = \frac{c^{(+)}}{2}\, e_2$ (AppendixA). Thus

$$\nabla^{C'}_{e_1} e_3 \;=\; \frac{c^{(+)}}{2}\, e_2 - \frac{c^{(+)} + \psi}{2}\, e_2 \;=\; -\frac{\psi}{2}\, e_2.$$

We compute the curvature $R^{C'}(e_1, e_3)e_3 \;=\; \nabla^{C'}_{e_1}\nabla^{C'}_{e_3} e_3 - \nabla^{C'}_{e_3}\nabla^{C'}_{e_1} e_3 - \nabla^{C'}_{[e_1,e_3]} e_3$. The first term vanishes because $\nabla^{C'}_{e_3} e_3 = 0$ (antisymmetry of T'). The third term vanishes because $[e_1, e_3] = 0$. For the second term,

$$\nabla^{C'}_{e_3}\left(-\frac{\psi}{2}\, e_2\right) \;=\; -\frac{e_3(\psi)}{2}\, e_2 \;-\; \frac{\psi}{2}\, \nabla^{C'}_{e_3} e_2.$$

Since ψ depends only on the base coordinates (k_x, k_y) (the modification is a form pulled back from T^2_{BZ}), we have $e_3(\psi) = \partial_{\phi_+}\psi = 0$. The covariant derivative $\nabla^{C'}_{e_3} e_2 = \nabla^{\mathrm{LC}(1)}_{e_3} e_2 + \frac{1}{2}T'_{e_3}(e_2)$ evaluates to $-\frac{2c^{(+)}+\psi}{2}\, e_1$ by a direct computation using the torsion endomorphism table (AppendixB) with the modified coefficient $c^{(+)} + \psi$. Combining,

$$R^{C'}(e_1, e_3)\, e_3 \;=\; -\frac{\psi}{2} \cdot \frac{2c^{(+)} + \psi}{2}\, e_1 \;=\; -\frac{\psi(2c^{(+)} + \psi)}{4}\, e_1.$$

On U_ψ, the function ψ is non-zero by definition, and the factor $2c^{(+)} + \psi$ is non-zero on the non-empty open subset $\{|2c^{(+)}| > |\psi|\} \cap U_\psi$ (which is non-empty for any ψ with sufficiently small L^∞ norm). The resulting operator maps $e_3 \in \mathcal{V}_p$ to $e_1 \in \mathcal{H}_p$, so it is purely off-diagonal. The Ambrose–Singer theorem yields $\dim \mathfrak{hol}^{\mathrm{off}}_p(\nabla^{C'}) \geq 1$.

Proof of (ii). Among torsion representatives of pure bidegree $(2, 1)$, the general form is $T' = (c^{(+)} + \psi)\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_+ + (c^{(-)} + \chi)\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_-$ with ψ, χ satisfying the cohomological constraint $[\psi\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_+ + \chi\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_-] = 0$. An analogous computation shows

$$\nabla^{C'}_{e_1} e_3 = -\frac{\psi}{2}\, e_2, \qquad \nabla^{C'}_{e_1} e_4 = -\frac{\chi}{2}\, e_2.$$

The off-diagonal curvature vanishes identically if and only if $\psi \equiv 0$ and $\chi \equiv 0$, which corresponds to $T' = T_{\mathrm{Bis}}$. This establishes uniqueness.

Proof of (iii). The computation in (4.5) with $\varepsilon = 1$ gives $R^{\mathrm{LC}(1)}(e_1, e_3)e_3 = \frac{(c^{(+)})^2}{4}\, e_1$, which is purely off-diagonal and non-zero for $c^{(+)} \neq 0$. By the Ambrose–Singer theorem applied to $\nabla^{\mathrm{LC}(1)}$, the Riemannian holonomy algebra $\mathfrak{hol}_p(\nabla^{\mathrm{LC}(1)})$ contains a non-zero off-diagonal element. □

Remark 4.8 (Explicit non-Bismut representative). *A concrete representative satisfying the hypotheses of Theorem4.7 is obtained as follows. Let $F(k_x, k_y) = \sin(k_x)$ and define $\alpha = F\, dk_y \wedge d\phi_+ \in \Omega^2(M)$. Then $d\alpha = \cos(k_x)\, \mathrm{vol}_{\mathrm{BZ}} \wedge d\phi_+$ is exact, and $T' := T_{\mathrm{Bis}} + d\alpha$ satisfies $[T'] = [T_{\mathrm{Bis}}] = [\omega]$ with $\psi(k_x, k_y) = \cos(k_x)$. The set $U_\psi = \{p \in M : \cos(k_x) \neq 0\}$ is open dense in M, and for $|c^{(+)}| > 1/2$ the factor $2c^{(+)} + \cos(k_x) \neq 0$ everywhere. In this case (4.11) is non-zero on all of U_ψ, and $\dim \mathfrak{hol}^{\mathrm{off}}_p(\nabla^{C'}) \geq 1$ on an open dense subset of M.*

4.4 Geometric and physical interpretation

The analysis of Sections4.2–4.3 reveals a three-level structure in the off-diagonal holonomy of the SOC BEC model.

At the Riemannian level, the Levi-Civita curvature of the physical Kaluza–Klein metric g_1 has non-trivial off-diagonal components (Theorem4.7(iii)): the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$ is irreducible for the Riemannian holonomy. This is a robust geometric feature that depends only on the non-vanishing of the Berry curvatures $F^{(\pm)}$ and persists under any smooth deformation of the metric within the Kaluza–Klein class.

At the torsion-connection level, the off-diagonal holonomy depends on the choice of torsion representative. The Bismut torsion—the unique representative for which the horizontal distribution is parallel along horizontal directions—yields reducible holonomy. Every other representative in the same cohomology class produces irreducible holonomy on an open non-empty subset (Theorem4.7(i)–(ii)).

At the cohomological level, the invariant $r^\sharp = 1$ correctly predicts that the mixed class $[\omega]$ is not absorbed by the parallel-form strata for the physical metric. The Bismut cancellation is an analytic phenomenon—the exact compensation of the LC and contorsion contributions—that is invisible to the cohomological invariant; it is a codimension-infinity phenomenon in the space of torsion representatives and is therefore non-generic.

Remark 4.9 (Topological locking of degrees of freedom)**.** *The off-diagonal holonomy—whether detected via the Riemannian curvature of g_1 or via a non-Bismut torsion representative—implies that the momentum and phase degrees of freedom are entangled by the mixed cohomology class $[\omega]$ in a way that cannot be undone by smooth gauge transformations or metric deformations preserving the cohomology classes $[F^{(\pm)}]$. This locking persists even when local gauge transformations can eliminate the Berry curvature in restricted subspaces of the parameter space.*

Remark 4.10 (Independence of the total Chern number)**.** *The bound remains non-trivial even when the total Chern number vanishes:*

$$c_1^{\mathrm{tot}} := c_1^{(+)} + c_1^{(-)} = \frac{1}{2\pi} \int_{T^2_{\mathrm{BZ}}} \left(F^{(+)} + F^{(-)} \right) = 0.$$

As long as the individual curvatures are non-zero ($c^{(\pm)} \neq 0$), the mixed rank remains $r = 1$ and the obstruction kernel vanishes for the physical metric, yielding $r^\sharp = 1$. The structure of the mixed class—not just its integral—determines the irreducibility. In particular, the Levi-Civita off-diagonal curvature (4.12) *depends on the individual constants $c^{(\pm)}$ and is insensitive to cancellations in their sum.*

Remark 4.11 (Robustness under perturbations)**.** *In a physical SOC BEC, perturbations such as interactions, disorder, and spatial inhomogeneities in the Raman coupling generically break the exact Bismut cancellation. More precisely, any perturbation that modifies the torsion 3-form by an amount $\psi \neq 0$—however small—while remaining within the same cohomology class, produces a non-Bismut representative for which* $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ *on an open set (Theorem4.7(i)). Since the Bismut condition $\psi \equiv 0$ has infinite codimension in the space of smooth torsion representatives, the topological obstruction certified by the bound is a robust and generic feature of the SOC BEC phase diagram.*

Remark 4.12 (Relation to the cohomological lower bound of[12])**.** *The lower bound of[12] applies to cohomologically calibrated connections on Riemannian product manifolds and yields the invariant* $r^\sharp = \operatorname{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}$*, where* $\operatorname{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big)$ *is the mixed tensor rank and* $\mathcal{K}$ *is the obstruction kernel determined by parallel-form strata. In the product limit* $\varepsilon = 0$ *of our model, the metric* g_0 *is a Riemannian product on* $T^2_{\mathrm{BZ}} \times T^2_{\mathrm{fiber}}$ *with both factors flat, so* $\mathcal{P}_1(T^2_{\mathrm{fiber}}) = \mathbb{R}^2$ *and* $r^\sharp = 0$*: the bound is trivial, consistently with the fact that the product geometry absorbs the mixed class into the parallel-form strata.*

For $\varepsilon > 0$*, the Kaluza–Klein metric* g_ε *is no longer a Riemannian product, and the theorem of [12] does not apply directly. Nevertheless, the algebraic mechanism that drives the transition* $r^\sharp = 0 \to r^\sharp = 1$ *in [12]—the collapse of the parallel-form stratum* $\mathcal{P}_1$ *from* $\mathbb{R}^2$ *to a proper subspace—has an exact counterpart in our setting: Theorem 2.14 shows that* $\dim \mathcal{P}_1^\varepsilon$ *drops from 2 to 1 as soon as* $\varepsilon > 0$*. For* $\varepsilon \in (0, 1)$*, the direct curvature computation of Theorem4.5 confirms that this drop is accompanied by the appearance of off-diagonal holonomy, as the cohomological framework would predict. At the Kaluza–Klein endpoint* $\varepsilon = 1$*, the Bismut cancellation suppresses the off-diagonal holonomy of the natural torsion, but not that of generic representatives of the same class (Theorem4.7), nor that of the Levi-Civita connection itself.*

The invariant $r^\sharp = 1$ *therefore captures the correct topological obstruction even at the KK endpoint: the obstruction is real, and its apparent absence for the Bismut torsion is an artefact of the precise balance between the LC and contorsion terms—a balance that is non-generic and broken by any perturbation of the torsion representative within the cohomology class.*

4.5 Summary of the proof strategy

The logical structure of the proof can be summarized as follows.

Step 1: Deformation family (Section2). We construct a smooth one-parameter family of metrics $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ interpolating between the Riemannian product metric $g_0 = g_{\mathrm{BZ}} \oplus d\phi_+^2 \oplus d\phi_-^2$ and the physical Kaluza–Klein metric $g_M = g_1$.

Step 2: Topological and geometric invariants (Section3). We compute the mixed tensor rank $r = \operatorname{rank}_{\mathbb{R}}([\omega]_{\mathrm{mixed}}) = 1$ and the obstruction kernels $\dim \mathcal{K}_0 = 1$ at $\varepsilon = 0$ and $\dim \mathcal{K}_\varepsilon = 0$ for $\varepsilon > 0$. This yields reduced ranks $r_0^\sharp = 0$ (trivial) and $r_\varepsilon^\sharp = 1$ (non-trivial).

Step 3: Product case ($\varepsilon = 0$). The PT theorem applies directly, giving the trivial bound $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_0}) \geq 0$.

Step 4: Exact curvature formula ($\varepsilon \in (0, 1)$). Through explicit computation of the curvature $R^{C_\varepsilon}(X, Z)$ for mixed inputs (Theorem4.5, AppendixB), we obtain the exact identity $R^{C_\varepsilon}(e_1, e_3)e_3 = \frac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_1$, establishing $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ for every $\varepsilon \in (0, 1)$ and every $c^{(+)} \neq 0$.

Step 5: Physical endpoint ($\varepsilon = 1$). The natural KK torsion T_{Bis} produces exact cancellation at $\varepsilon = 1$ (the Bismut phenomenon). This cancellation is characterised as non-generic: every non-Bismut representative of $[\omega]$ yields $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ on an open non-empty set (Theorem4.7(i)), and the Levi-Civita curvature of g_1 retains non-trivial off-diagonal components (Theorem4.7(iii)).

This strategy rigorously applies the PT cohomological framework to the specific Kaluza–Klein geometry of the SOC BEC model. The essential topological constraint—encoded in $r_\varepsilon^\sharp = 1$ for $\varepsilon > 0$—translates into a geometric lower bound on off-diagonal holonomy for all $\varepsilon \in (0, 1)$ and, at the KK endpoint $\varepsilon = 1$, for all non-Bismut torsion representatives. The invariant $r^\sharp$ thus provides a cohomological certificate for locally irremovable curvature in the SOC BEC parameter space, operating beyond the Chern-number paradigm.

5 Topological Lower Bound and Its Physical Consequences

With the geometric construction of Section2, the cohomological data of Section3, and the curvature analysis of Section4 at hand, we now synthesize these results to obtain the main physical consequence: a topological lower bound on the off-diagonal holonomy of the synthetic gauge connection in the SOC BEC model, together with a complete characterisation of the Bismut cancellation at the Kaluza–Klein endpoint.

5.1 Statement of the bound

The central results are expressed by two complementary inequalities. The first holds for the deformation family away from the KK endpoint:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \;\geq\; r^\sharp := \mathrm{rank}_{\mathbb{R}}\big([\omega]_{\mathrm{mixed}}\big) - \dim \mathcal{K}_\varepsilon \;=\; 1 \qquad \text{for all } \varepsilon \in (0,1). \tag{5.1}$$

The second holds at the physical KK metric g_1 for any non-Bismut torsion representative:

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C'}) \;\geq\; 1 \qquad \text{on an open non-empty subset of } M, \tag{5.2}$$

for every cohomologically calibrated connection $\nabla^{C'} = \nabla^{\mathrm{LC}(1)} + \frac{1}{2}T'$ with $[T'] = [\omega]$ and $T' \neq T_{\mathrm{Bis}}$.

5.2 Main theorem: topological lower bound for SOC BECs

Theorem 5.1 (Topological lower bound for SOC BECs)**.** *Let $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$ be the parameter space equipped with the Kaluza–Klein metric g_1 (Definition2.5). Under Assumption2.3 (constant Berry curvatures with $c^{(\pm)} \neq 0$):*

(i) *(Deformation family.) For every $\varepsilon \in (0,1)$, the metric connection ∇^{C_ε} with the natural Kaluza–Klein torsion satisfies*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \;\geq\; 1 \qquad \textit{at every point } p \in M.$$

(ii) *(Physical metric, non-Bismut torsion.) For any cohomologically calibrated connection $\nabla^{C'} = \nabla^{\mathrm{LC}(1)} + \frac{1}{2}T'$ on (M, g_1) with $[T'] = [\omega]$ and $T' \neq T_{\mathrm{Bis}}$,*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C'}) \;\geq\; 1 \qquad \textit{on an open non-empty subset of } M.$$

(iii) *(Physical metric, Riemannian holonomy.) The Levi-Civita connection of g_1 satisfies*

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \;\geq\; 1 \qquad \textit{at every point } p \in M.$$

(iv) *(Bismut uniqueness.) The Bismut torsion $T_{\mathrm{Bis}} = F^{(+)} \wedge \Theta^{(+)} + F^{(-)} \wedge \Theta^{(-)}$ is the unique torsion representative of $[\omega]$ of pure bidegree $(2,1)$ for which $\mathfrak{hol}^{\mathrm{off}}_p(\nabla^C) = \{0\}$ at every point.*

In all cases, the cohomological invariant satisfies $r^\sharp = 1$ for the physical metric.

Proof. Part (i) follows from Theorem4.5: the exact formula $R^{C_\varepsilon}(e_1, e_3)e_3 = \frac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_1$ is non-zero for $\varepsilon \in (0, 1)$, and the Ambrose–Singer theorem yields the bound at every point by homogeneity.

Parts (ii) and (iv) follow from Theorem4.7: the non-Bismut curvature (4.11) is non-zero on $U_\psi = \{\psi \neq 0\}$, which is open and non-empty for any non-trivial modification; uniqueness of the Bismut cancellation is established in Theorem4.7(ii).

Part (iii) follows from Theorem4.7(iii): the Levi-Civita curvature $R^{\mathrm{LC}(1)}(e_1, e_3)e_3 = \frac{(c^{(+)})^2}{4}\, e_1 \neq 0$ is off-diagonal and constant on M. □

Remark 5.2 (Methodological scope)**.** *The adaptation strategy employed in this work does not rely on specific features of the SOC BEC model beyond the constant curvature assumption (Assumption2.3). The key ingredients are:*

1. *a smooth deformation family $\{g_\varepsilon\}_{\varepsilon\in[0,1]}$ interpolating between a Riemannian product metric ($\varepsilon = 0$) and the physical metric ($\varepsilon = 1$);*
2. *explicit computation of the parallel-form strata $\mathcal{P}_k(M_i, g_\varepsilon)$ for the deformed metrics;*
3. *verification that the obstruction kernel $\mathcal{K}_\varepsilon$ has reduced dimension for $\varepsilon > 0$ compared to the product limit;*
4. *identification of the Bismut cancellation at the KK endpoint and its characterisation as a non-generic phenomenon.*

This framework applies whenever the total space admits a fibered structure with computable cohomology and parallel forms. Natural candidates include Kaluza–Klein reductions in supergravity, principal bundles with connection in Yang–Mills theory, and synthetic gauge field systems in photonics and connon-emptyd matter physics.

5.3 Physical interpretation: three levels of irreducibility

The analysis of Section4 reveals a three-level structure in the off-diagonal holonomy of the SOC BEC model:

1. *Riemannian level.* The Levi-Civita curvature of g_1 has non-trivial off-diagonal components (Theorem5.1(iii)): the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$ is irreducible for the Riemannian holonomy. This is a robust geometric feature that depends only on the non-vanishing of the Berry curvatures $F^{(\pm)}$.
2. *Torsion-connection level.* For non-Bismut torsion representatives, the off-diagonal holonomy is non-trivial on an open non-empty set (Theorem5.1(ii)). The Bismut torsion is the unique exception (Theorem5.1(iv)).
3. *Cohomological level.* The invariant $r^\sharp = 1$ correctly predicts that the mixed class $[\omega]$ is not absorbed by the parallel-form strata. The Bismut cancellation is an analytic phenomenon of infinite codimension in the space of torsion representatives, invisible to the cohomological invariant.

At least one independent off-diagonal curvature operator therefore persists at every level, mixing momentum with the phase directions $\phi_\pm$. This represents a topological obstruction to simultaneous flattening of the Berry curvature along both phase circles that cannot be removed by any smooth gauge transformation or metric deformation preserving the cohomology classes $[F^{(\pm)}]$.

The refined invariant $r^\sharp$ captures an operationally distinct feature from the mixed rank r. While $r = 1$ guarantees off-diagonal holonomy for any metric (Corollary 5.3 of [12]), it does not distinguish whether the obstruction can be circumvented by freezing a single linear combination of the two phases. At the product metric ($r^\sharp = 0$), the harmonic representative $\beta = c^{(+)}d\varphi_+ + c^{(-)}d\varphi_-$ belongs to the parallel-form stratum $\mathcal{P}_1$, and dimensional reduction along the corresponding circle S^1_β—physically, locking the phase combination $c^{(+)}\varphi_+ + c^{(-)}\varphi_-$—annihilates the torsion on the reduced space, eliminating the topological obstruction. For the Kaluza–Klein metric ($r^\sharp = 1$), the parallel-form stratum has rotated to be orthogonal to β: no single phase-locking protocol can eliminate the obstruction. The invariant $r^\sharp$ thus detects the robustness of the topological constraint under phase-reduction protocols, a distinction invisible to r alone.

5.4 Persistence of the bound when the total Chern number vanishes

A crucial feature is that the bound is insensitive to the total first Chern number $c_1^{\mathrm{tot}} = c_1^{(+)} + c_1^{(-)}$.

Corollary 5.3 (Bound independent of total Chern number)**.** *If $c_1^{(+)} \neq 0$ and $c_1^{(-)} = -c_1^{(+)}$ (so $c_1^{\mathrm{tot}} = 0$), all four parts of Theorem5.1 remain valid. In particular:*

(a) *For $\varepsilon \in (0,1)$, $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ at every point.*

(b) *For the physical metric with any non-Bismut torsion, $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C'}) \geq 1$ on an open non-empty set.*

(c) *The Riemannian holonomy of g_1 is irreducible: $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \geq 1$ at every point.*

Thus the topological obstruction persists even when the net Berry flux vanishes.

Proof. As long as both $F^{(\pm)} \neq 0$, the mixed tensor rank is $r = 1$ and $\mathcal{K}_\varepsilon = 0$ for $\varepsilon > 0$ (Lemma2.17). The Levi-Civita off-diagonal curvature (4.12) depends on the individual constants $c^{(\pm)}$ and is insensitive to cancellations in their sum. The bound follows from Theorem5.1. □

5.5 Robustness and experimental signatures

The bound depends on topological data ($[\omega]$ and parallel-form strata) and on the non-genericity of the Bismut cancellation, making it robust against perturbations. In a physical SOC BEC, perturbations such as interactions, disorder, and spatial inhomogeneities in the Raman coupling generically break the exact Bismut cancellation, placing the system in the regime of Theorem5.1(ii).

Proposition 5.4 (Experimental signature)**.** *Under Assumption2.3, no smooth gauge transformation can simultaneously eliminate the Berry phases* Φ_B^+ *and* Φ_B^- *accumulated along any non-contractible loop* $\gamma \subset T^2_{\mathrm{BZ}}$*. Even when the total Chern number vanishes, the persistent off-diagonal curvature—at the Riemannian level or at the torsion-connection level for non-Bismut representatives—guarantees at least one non-zero component of* $(\Phi_B^+(\gamma), \Phi_B^-(\gamma))$ *for every* γ*.*

5.6 Summary

Theorem5.1 provides a cohomologically defined obstruction to complete Berry curvature removal in SOC BECs, valid even when global Chern invariants vanish. The obstruction manifests at three levels: the Riemannian holonomy of the KK metric is always irreducible; the holonomy of the torsion connection is irreducible for every non-Bismut representative; and the Bismut cancellation is characterised as the unique, non-generic exception. The invariant $r^\sharp = 1$ provides a cohomological certificate for this obstruction, extending the Chern-number paradigm to a finer classification based on mixed cohomology classes.

6 Physical Consequences and Experimental Signatures

The three-level irreducibility structure established in Theorem5.1 has direct physical consequences for the synthetic gauge structure of spin-orbit-coupled BECs. These consequences manifest as irreducible geometric couplings between momentum and phase degrees of freedom, with observable signatures in interferometric measurements.

6.1 Geometric obstruction to gauge flattening

The Riemannian holonomy bound $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \geq 1$ (Theorem5.1(iii)) implies that the curvature tensor of the physical Kaluza–Klein metric cannot be made block-diagonal with respect to the splitting $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$. At the torsion-connection level, the same conclusion holds for every non-Bismut representative of $[\omega]$ on an open non-empty subset (Theorem5.1(ii)). In more physical terms, the Riemannian geometry of the extended parameter space irreducibly couples momentum to phase directions, and this coupling is reinforced at the torsion-connection level for generic torsion representatives.

Definition 6.1 (Mixed curvature components)**.** *Let* $X \in \mathcal{H}_p$ *be a horizontal vector (momentum direction) and* $Z \in \mathcal{V}_p$ *a vertical vector (phase direction). The mixed curvature components are defined as:*

$$\Omega(X, Z) := R(X, Z) \in \mathfrak{so}(T_pM),$$

where R *denotes the curvature of either* $\nabla^{\mathrm{LC}(1)}$ *or* $\nabla^{C'}$ *(for a non-Bismut representative). These endomorphisms map horizontal directions to vertical ones and vice versa.*

Theorem5.1(iii) guarantees that at least one such mixed component of $R^{\mathrm{LC}(1)}$ is non-zero at every point $p \in M$. For non-Bismut torsion representatives, Theorem5.1(ii) provides the same guarantee on an open non-empty subset. This represents a topological obstruction to flattening the synthetic gauge connection simultaneously along both phase circles ϕ_+ and ϕ_-.

Remark 6.2 (The Bismut exception)**.** *For the Bismut torsion* T_{Bis}*, the mixed curvature components of* ∇^{C_1} *vanish identically (Section4.3). However, the Riemannian mixed curvature* $R^{\mathrm{LC}(1)}(X, Z)$ *remains non-zero, so the geometric coupling between momentum and phase directions persists at the Riemannian level regardless of the torsion representative. The Bismut cancellation is a non-generic phenomenon (Theorem4.7(ii)) that is broken by any physical perturbation modifying the torsion within its cohomology class.*

6.2 Interpretation in terms of Berry curvature non-integrability

The mixed holonomy bound has a direct interpretation in terms of the non-integrability of the Berry connection. For a two-component SOC BEC, the synthetic gauge fields $A^{(\pm)}$ give rise to Berry curvatures $F^{(\pm)} = dA^{(\pm)}$. The topological lower bound implies that even if the net Chern number vanishes, the Berry connection cannot be simultaneously made flat in both $U(1)$ sectors.

Corollary 6.3 (Non-flattenability of Berry curvature)**.** *Under Assumption2.3, suppose* $c^{(+)} \neq 0$ *and* $c^{(-)} \neq 0$*. Then there does not exist a smooth gauge transformation*

$$A^{(\pm)} \mapsto A^{(\pm)} + d\lambda^{(\pm)}$$

that makes both curvatures vanish identically, i.e., such that $F^{(+)} = F^{(-)} = 0$ *everywhere on* T^2_{BZ}*.*

Proof. The curvatures $F^{(\pm)} = dA^{(\pm)}$ are gauge-invariant 2-forms: under $A^{(\pm)} \mapsto A^{(\pm)} + d\lambda^{(\pm)}$, the curvatures are unchanged ($F^{(\pm)} \mapsto F^{(\pm)}$). Since $c^{(\pm)} \neq 0$, we have $F^{(\pm)} \neq 0$ in every gauge. More fundamentally, the cohomology classes $[F^{(\pm)}] \in H^2(T^2_{\mathrm{BZ}}; \mathbb{R})$ are non-zero topological invariants that cannot be altered by gauge transformations. □

This result is stronger than the usual Chern number obstruction: the Chern number only prevents making the curvature exact, but here we show that even when the Chern numbers cancel ($c_1^{(+)} = -c_1^{(-)}$), the curvature cannot be made to vanish pointwise in both sectors simultaneously. The geometric origin of this obstruction is the Riemannian irreducibility of the KK metric (Theorem5.1(iii)), which depends on the individual curvatures $c^{(\pm)}$ rather than their sum.

6.3 Experimental signatures via interferometry

The geometric obstruction predicted by Theorem5.1 suggests concrete interferometric tests. Consider measuring the Berry phases accumulated along closed loops $\gamma \subset T^2_{\mathrm{BZ}}$ separately for the two phase sectors ϕ_+ and ϕ_-.

Definition 6.4 (Two-component Berry phase vector)**.** *For a closed loop* γ *in the Brillouin zone, define:*

$$\Phi_B(\gamma) = \begin{pmatrix} \Phi_B^+(\gamma) \\ \Phi_B^-(\gamma) \end{pmatrix},$$

where $\Phi_B^{\pm}(\gamma) = \oint_\gamma A^{(\pm)}$ *are the Berry phases for the two* $U(1)$ *sectors.*

If the synthetic gauge field were fully reducible (i.e., if the holonomy were block-diagonal), one could choose gauges where both $\Phi^+_B(\gamma)$ and $\Phi^-_B(\gamma)$ vanish for every loop γ. The Riemannian irreducibility bound $\dim \mathfrak{hol}^{\text{off}}(\nabla^{\text{LC}(1)}) \geq 1$ forbids such simultaneous vanishing, providing the most robust experimental prediction: it holds at every point of M and is independent of the choice of torsion representative. For non-Bismut torsion connections, the torsion-level bound $\dim \mathfrak{hol}^{\text{off}}(\nabla^{C'}) \geq 1$ provides an additional, generically satisfied constraint.

6.4 Comparison with Chern number physics

The mixed-rank invariant $r^\sharp$ provides a finer classification than the Chern number in several important respects:

Chern number	*Mixed rank invariant* $r^\sharp$
Global invariant (integrated over BZ)	Local structural invariant
Quantizes net vorticity	Quantifies irreducible coupling
Vanishes when fluxes cancel	Non-zero when individual fluxes are non-zero
Detects topological charge	Detects topological entanglement

Table 1: Comparison between Chern number and mixed-rank invariant.

The key distinction is that the Chern number c_1^{tot} can vanish due to cancellation of local contributions, while $r^\sharp$ remains non-zero as long as both individual curvatures are non-vanishing. This makes $r^\sharp$ particularly valuable for detecting topological effects in compensated systems or synthetic vacuum configurations.

6.5 Robustness and experimental feasibility

The topological lower bound is robust against typical experimental perturbations for three reasons:

1. *Topological invariance*: The mixed tensor rank r depends only on cohomology classes, which are stable under small deformations of the Hamiltonian.

2. *Riemannian robustness*: The Levi-Civita off-diagonal curvature (Theorem5.1(iii)) depends only on the non-vanishing of $c^{(\pm)}$ and is independent of the torsion representative. This provides a prediction that is immune to the Bismut cancellation.

3. *Generic non-Bismut regime*: Physical perturbations (interactions, disorder, spatial inhomogeneities) generically break the exact Bismut cancellation, placing the system in the regime of Theorem5.1(ii) where the torsion-connection bound is also non-trivial.

Current experimental capabilities in cold-atom systems are sufficient to test these predictions:

- *Independent phase control*: Optical "painting" techniques [8] allow independent manipulation of ϕ_+ and ϕ_-.

- *Berry phase measurement*: Interferometric methods used to map Berry curvature in Hofstadter bands [2] can be adapted to track two phase sectors.

- *Momentum-space tomography*: Time-of-flight measurements provide access to the Brillouin zone geometry.

A proposed experiment would:

1. Prepare a two-component SOC BEC in a toroidal trap.
2. Adiabatically transport the condensate along independent cycles γ_1, γ_2 in the Brillouin zone.
3. Measure the accumulated phases Φ_B^+ and Φ_B^- separately using interferometry.
4. Verify that no gauge transformation can simultaneously eliminate both phases.

6.6 Implications for topological quantum matter

The results presented here extend the classification of synthetic gauge fields beyond conventional topological invariants. The mixed-rank framework:

1. *Reveals hidden topology*: Detects topological obstructions invisible to Chern numbers, operating at the level of mixed cohomology classes rather than integrated curvature.
2. *Provides a hierarchy of constraints*: The three-level irreducibility structure (Riemannian, non-Bismut torsion, deformation family) offers complementary geometric perspectives on the same topological obstruction, with the Riemannian level providing the most robust prediction.
3. *Generalizes to multi-component systems*: The approach naturally extends to systems with more than two internal states, where the extended parameter space acquires additional $U(1)$ factors and the mixed tensor rank can exceed 1.

This suggests that mixed cohomology classes provide a powerful tool for characterizing the topological structure of synthetic gauge fields in engineered quantum systems, with potential applications beyond cold atoms to photonic systems, superconducting circuits, and other synthetic quantum platforms.

7 Illustrative Examples: From Chern Numbers to the Mixed Rank

The three-level irreducibility structure established in Theorem5.1 provides a universal constraint for generic two-component SOC BECs. In this section, we illustrate its operational significance in two paradigmatic scenarios: first, the standard Rashba–Dresselhaus texture, where our framework recovers the phenomenology of Chern-number-induced obstructions; second, a configuration with vanishing net Chern flux, where the mixed-rank invariant $r^\sharp$ reveals topological features inaccessible to traditional global invariants.

7.1 Example 1: Rashba–Dresselhaus texture and the maximal rank $r = 1$

Consider a two-dimensional, two-level Hamiltonian of the Rashba–Dresselhaus (RD) type:

$$\hat{H}_{\mathrm{RD}}(k) \;=\; \frac{1}{2m}\big(\mathbf{p} - \mathbf{A}_{\mathrm{syn}}(k)\big)^2 \;+\; \mathbf{B}_{\mathrm{syn}}(k)\cdot\boldsymbol{\sigma}, \qquad k = (k_x, k_y) \in T^2_{\mathrm{BZ}},$$

where $\mathbf{A}_{\mathrm{syn}}(k)$ and $\mathbf{B}_{\mathrm{syn}}(k)$ represent the synthetic vector potential and Zeeman field, respectively. For generic coupling parameters κ and Ω, the lower dressed band induces a spin texture map $\mathbf{n} : T^2_{\mathrm{BZ}} \to S^2$.

The associated eigenline bundle $\mathbb{L} \to T^2_{\mathrm{BZ}}$ has a first Chern number $c_1(\mathbb{L})$ equal to the degree of $\mathbf{n}$. For a minimal RD texture with degree 1, the Berry curvature F represents a non-trivial cohomology class $[F] \in H^2(T^2_{\mathrm{BZ}}; \mathbb{Z})$.

In the extended parameter space $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, the synthetic torsion class is

$$[\omega] = [F] \otimes [d\phi_+] \;+\; [F] \otimes [d\phi_-] \in H^3(M; \mathbb{R}).$$

Since $\dim H^2(T^2_{\mathrm{BZ}}) = 1$, the two summands are linearly dependent, and the class factors as a single simple tensor:

$$[\omega] = [F] \otimes \big([d\phi_+] + [d\phi_-]\big).$$

Hence, the mixed tensor rank is $r = 1$. For the product metric g_0, the obstruction kernel has $\dim \mathcal{K}_0 = 1$, giving $r^\sharp_0 = 0$. However, for the physical Kaluza–Klein metric g_M (or any g_ε with $\varepsilon > 0$), Theorem3.14 shows $\mathcal{K} = 0$, so $r^\sharp = 1$. Theorem5.1 then yields the three-level structure:

(i) For $\varepsilon \in (0, 1)$: $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ at every point.

(ii) At $\varepsilon = 1$ with any non-Bismut torsion: $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C'}) \geq 1$ on an open non-empty set.

(iii) The Riemannian holonomy of g_1 satisfies $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \geq 1$ at every point.

Remark 7.1 (Local vs. global structure)**.** *When $c_1(\mathbb{L}) \neq 0$, the underlying principal bundle is topologically non-trivial. Nevertheless, our geometric analysis is performed in local trivializations, where M is diffeomorphic to T^4 and the metric takes the form* (2.1)*. The deformation argument and the pointwise curvature computation are local in nature and can be applied in each trivialization. Thus, the bounds hold locally, detecting the irreducible coupling between momentum and phase directions even when the global bundle is non-trivial.*

7.2 Example 2: Topological persistence under vanishing net Chern flux

The power of the mixed-rank invariant becomes most apparent when global topological charges cancel. Consider a configuration where the two $U(1)$ sectors carry Berry curvatures with equal and opposite integrated fluxes:

$$\frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F^{(+)} = +1, \qquad \frac{1}{2\pi}\int_{T^2_{\mathrm{BZ}}} F^{(-)} = -1. \tag{7.1}$$

Then the total Chern number $c_1^{\mathrm{tot}} = c_1^{(+)} + c_1^{(-)} = 0$. Conventional Chern-number diagnostics would classify this regime as topologically trivial, yet our bound reveals a residual topological obstruction.

Under Assumption2.3 we have $F^{(\pm)} = c^{(\pm)}\,\mathrm{vol}_{\mathrm{BZ}}$ with $c^{(+)} = -c^{(-)}$. The cohomology classes satisfy $[F^{(-)}] = -[F^{(+)}]$. Hence, the torsion class becomes

$$[\omega] = [F^{(+)}] \otimes [d\phi_+] + [F^{(-)}] \otimes [d\phi_-] = [F^{(+)}] \otimes \big([d\phi_+] - [d\phi_-]\big),$$

which again has mixed rank $r = 1$. Since $F^{(\pm)} \neq 0$, the obstruction kernel vanishes for the physical metric, giving $r^\sharp = 1$.

We now illustrate the three-level irreducibility structure concretely. Choose the explicit connections

$$A^{(+)} = \frac{1}{4\pi}(k_x dk_y - k_y dk_x), \qquad A^{(-)} = -A^{(+)},$$

which satisfy (7.1) on a torus of period 2π in each direction. Write $c := c^{(+)} = 1/(2\pi)$, so that $c^{(-)} = -c$. The torsion 3-form in adapted gauge is

$$T = c\, dk_x \wedge dk_y \wedge (d\phi_+ - d\phi_-).$$

Level 1: Deformation family ($\varepsilon \in (0, 1)$)**.** The exact curvature formula of Theorem4.5 gives, for the mixed pair (e_1, e_3):

$$R^{C_\varepsilon}(e_1, e_3)\, e_3 = \frac{c^2(\varepsilon^2 - 1)}{4}\, e_1.$$

For $\varepsilon \in (0, 1)$, the factor $(\varepsilon^2-1) < 0$ is non-zero, confirming $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ at every point. Analogously, all four mixed-input pairs (e_i, e_α) produce off-diagonal curvature proportional to $(\varepsilon^2 - 1)$ (PropositionB.11).

Level 2: Bismut cancellation at $\varepsilon = 1$. At $\varepsilon = 1$, the factor $(\varepsilon^2 - 1)$ vanishes exactly: $R^{C_1}(e_1, e_3) = 0$ for all basis vectors. This is the Bismut cancellation (Section4.3). The natural KK torsion $T_{\mathrm{Bis}} = c\,\mathrm{vol}_{\mathrm{BZ}} \wedge (d\phi_+ - d\phi_-)$ is the unique representative of $[\omega]$ of pure bidegree $(2, 1)$ for which this cancellation occurs (Theorem4.7(ii)).

Choosing instead the non-Bismut representative $T' = T_{\mathrm{Bis}} + d\alpha$ with $\alpha = \sin(k_x)\, dk_y \wedge d\phi_+$ (so that $\psi(k_x) = \cos(k_x)$), Theorem4.7(i) gives

$$R^{C'}(e_1, e_3)\, e_3\big|_p = -\frac{\cos(k_x)\big(2c + \cos(k_x)\big)}{4}\, e_1 \neq 0$$

on the open dense set $\{p \in M : \cos(k_x) \neq 0\}$, yielding $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C'}) \geq 1$.

Level 3: Riemannian holonomy of g_1. The Levi-Civita curvature of the physical KK metric is (Theorem4.7(iii)):

$$R^{\mathrm{LC}(1)}(e_1, e_3)\, e_3 = \frac{c^2}{4}\, e_1 = \frac{1}{16\pi^2}\, e_1 \neq 0.$$

This is purely off-diagonal ($\mathcal{V} \to \mathcal{H}$) and constant on M, certifying $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \geq 1$ at every point. The analogous computation for the pair (e_2, e_3) yields $R^{\mathrm{LC}(1)}(e_2, e_3)\, e_3 = \frac{c^2}{4}\, e_2$, providing a second linearly independent off-diagonal operator, so that

$$\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{\mathrm{LC}(1)}) \geq 2.$$

The Riemannian holonomy of the physical metric thus exceeds the cohomological lower bound $r^\sharp = 1$, even in this regime with vanishing total Chern number.

Remark 7.2 (Sharpness of the bound)**.** *The explicit computation confirms that the bound* $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ *is sharp in the sense that it cannot be improved without additional assumptions: there exist geometries (like the product metric* g_0*) where the off-diagonal holonomy is trivial. For generic Kaluza–Klein metrics, the bound is non-trivial and often exceeded: in the example above, the Riemannian holonomy alone generates a* ≥ 2*-dimensional off-diagonal algebra. The invariant* $r^\sharp = 1$ *is a lower bound; particular geometries exhibit a richer off-diagonal holonomy.*

Remark 7.3 (Role of the Bismut cancellation in examples)**.** *The vanishing of the off-diagonal curvature at* $\varepsilon = 1$ *for the Bismut torsion is a structural feature, not a pathology. It reflects the geometric fact that the natural KK connection makes horizontal parallel transport preserve the fiber directions—a property that defines the Bismut torsion. The three-level analysis shows that the topological obstruction encoded in* $r^\sharp = 1$ *manifests at every level except for this single, non-generic representative. In a physical SOC BEC, perturbations generically break the Bismut condition, placing the system in the regime of Level 2 above.*

7.3 Comparison: Global invariants vs. mixed rank

The distinction between c_1^{tot} and $r^\sharp$ reflects a fundamental dichotomy between integrated and structural topology:

Chern number c_1^{tot}	*Mixed rank* $r^\sharp$
Global aggregate of curvature	Local algebraic measure of coupling
Quantifies net vorticity	Quantifies irreducible entanglement
Can vanish by cancellation	Non-zero if each factor curvature is non-zero
Detects topological charge	Detects topological locking

Table 2: Comparison of topological invariants.

The mixed rank r counts the minimal number of simple tensors needed to represent the cohomology class $[\omega]$ in the Künneth decomposition. In our geometry, $r = 1$ whenever both individual Berry curvature classes $[F^{(\pm)}]$ are non-zero (and hence proportional, because $\dim H^2(T^2) = 1$). Thus r measures the irreducible geometric coupling between the Brillouin-zone torus and the phase fibres. The bound $\dim \mathfrak{hol}^{\mathrm{off}} \geq r^\sharp$ therefore constrains the algebraic complexity of the curvature, ensuring that the synthetic gauge field remains "locked" in a non-trivial configuration even when the net topological charge vanishes.

7.4 Summary

The Rashba–Dresselhaus example shows that the mixed-rank framework is consistent with established Chern-number results, while the $c_1^{\mathrm{tot}} = 0$ example demonstrates that it provides a finer classification, detecting locally non-removable curvature that conventional global invariants miss. The three-level analysis reveals that the topological obstruction encoded in $r^\sharp = 1$ persists at the Riemannian level (always), at the torsion-connection level (for all non-Bismut representatives), and throughout the deformation family (for $\varepsilon \in (0, 1)$). This makes $r^\sharp$ a robust tool for analyzing the stability of synthetic gauge fields in complex SOC BEC architectures, especially in compensated regimes where global topological charges cancel but local geometric obstructions persist.

8 Beyond the Chern Number: Local Geometric Constraints and Prospective Experimental Protocols

While the Chern number c_1 quantifies the net vorticity through a global integral over the Brillouin zone, the invariant $r^\sharp$ acts as a local structural constraint. It obstructs the simultaneous annihilation of Berry curvature components along independent directions within the product geometry. This phenomenon can be visualized as the Berry curvature being constrained to move along topological "rail-tracks" in the configuration space: even when the total topological charge is tuned to zero, specific directional components remain geometrically locked and cannot be erased by any smooth gauge transformation.

A concrete manifestation of this locking is the following. For any pair of independent cycles $\{\gamma_1, \gamma_2\}$ in T^2_{BZ}, the two-component Berry-phase vector

$$\mathbf{\Phi}_B = \big(\Phi_B^+(\gamma_1),\ \Phi_B^-(\gamma_1),\ \Phi_B^+(\gamma_2),\ \Phi_B^-(\gamma_2)\big)$$

cannot be continuously deformed to zero while preserving the mixed cohomology class $[\omega]$. This is a direct consequence of the Riemannian irreducibility of the physical Kaluza–Klein metric (Theorem5.1(iii)), which holds at every point of M and is independent of the torsion representative. Observing that these Berry-phase components cannot be simultaneously flattened would provide the first direct experimental signature of the topological obstruction captured by $r^\sharp$, effectively moving the study of quantum gases beyond the global Chern-number paradigm.

9 Conclusions

We have demonstrated that a two-component spin–orbit-coupled (SOC) BEC possesses an intrinsic geometric obstruction that prevents the simultaneous flattening of Berry curvature along the independent phase directions $\phi_\pm$. This obstruction persists even when the net global Chern flux vanishes, identifying a topological regime that eludes traditional integrated invariants.

The extended parameter space $M \cong T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$, equipped with the physical Kaluza–Klein metric g_M induced by synthetic gauge fields, carries a metric connection ∇^C with totally skew-symmetric torsion. Under Assumption2.3 (constant Berry curvatures), we have established that the mixed tensor rank of the torsion class is $r = 1$.

The core of our proof lies in the resolution of the Kaluza–Klein kernel problem and the identification of the Bismut cancellation. By introducing a smooth deformation family of metrics g_ε (Definition2.5), we have shown that the obstruction kernel $\mathcal{K}_\varepsilon$ vanishes for any $\varepsilon > 0$ due to the non-parallelism of the vertical coframe (Theorem2.14), yielding the cohomological invariant $r^\sharp = 1$.

The exact curvature formula $R^{C_\varepsilon}(e_1, e_3)\, e_3 = \frac{(c^{(+)})^2(\varepsilon^2-1)}{4}\, e_1$ (Theorem4.5) reveals a three-level irreducibility structure at the physical metric (Theorem5.1):

1. *Deformation family.* For $\varepsilon \in (0,1)$, the factor $(\varepsilon^2 - 1) \neq 0$ yields $\dim \mathfrak{hol}^{\mathrm{off}}(\nabla^{C_\varepsilon}) \geq 1$ at every point of M.

2. *Non-Bismut torsion at the physical metric.* At $\varepsilon = 1$, the natural Kaluza–Klein torsion—identified as the Bismut torsion—produces an exact cancellation of the off-diagonal curvature: the Levi-Civita and contorsion contributions compensate exactly, rendering horizontal parallel transport block-diagonal. This cancellation is characterised as the unique, non-generic exception: every other torsion representative of $[\omega]$ yields $\dim \mathfrak{hol}^{\mathrm{off}} \geq 1$ on an open non-empty subset (Theorem4.7(i)–(ii)).

3. *Riemannian holonomy.* The Levi-Civita curvature of the physical metric retains non-trivial off-diagonal components—$R^{\mathrm{LC}(1)}(e_1, e_3)\, e_3 = \frac{(c^{(+)})^2}{4}\, e_1 \neq 0$—certifying that the Riemannian holonomy is irreducible at every point, independently of the torsion representative.

The identification of the Bismut cancellation is both a subtlety and a strength of the analysis. The cancellation is a well-known structural feature of Kaluza–Klein geometry: the Bismut torsion is the unique totally skew-symmetric torsion for which the horizontal distribution of a principal bundle is parallel along horizontal directions. Its occurrence at $\varepsilon = 1$ is not a failure of the cohomological invariant $r^\sharp$—which correctly predicts the topological obstruction—but rather an analytic phenomenon of infinite codimension in the space of torsion representatives. In a physical SOC BEC, perturbations such as interactions, disorder, and spatial inhomogeneities in the Raman coupling generically break this exact balance, placing the system in the non-Bismut regime where the off-diagonal holonomy bound is non-trivial.

The persistence of the obstruction in regimes with zero total Chern number—as illustrated in Section7—highlights that the mixed-rank invariant $r^\sharp$ operates beyond the Chern-number paradigm, detecting structural features of the gauge field that integrated invariants fail to resolve. The three-level analysis shows that this obstruction manifests at every level of the geometric hierarchy: the Riemannian holonomy is always irreducible, the torsion-connection holonomy is irreducible for all non-Bismut representatives, and the deformation family exhibits irreducibility throughout $(0, 1)$.

Experimentally, this framework suggests interferometric protocols to resolve independent Berry phases (Section8), providing a direct signature of locally irremovable curvature. The Riemannian irreducibility (Level 3) provides the most robust prediction, as it depends only on the non-vanishing of the individual Berry curvatures and is independent of the choice of torsion representative.

Beyond the specific application to spin–orbit-coupled BECs, the methodology developed here—reducing a Kaluza–Klein geometry to its product limit while tracking the evolution of the obstruction kernel, deriving the exact curvature formula as a function of the deformation parameter, and characterising the Bismut cancellation at the KK endpoint—provides a general template for extending the Pigazzini–Toda lower bound [12] to fibered geometries beyond the Riemannian product setting. The essential requirements are that the parallel-form strata be computable for the physical metric and that the Bismut locus be identifiable within the space of torsion representatives. We expect this approach to be applicable to other physical systems where synthetic gauge fields induce non-trivial geometric phases, including photonic topological insulators, superconducting qubit arrays, and higher-dimensional Kaluza–Klein compactifications in string theory. A natural direction for future work is to investigate whether the Bismut cancellation admits physical counterparts in these systems and, if so, whether it corresponds to a fine-tuned regime with observable signatures.

Acknowledgements

The authors gratefully acknowledge stimulating discussions with colleagues in geometry and cold-atom physics that helped shape the formulation of these results.

A Levi-Civita Connection of the Deformed Metric

In this appendix, we derive the connection 1-forms and the covariant derivatives for the deformation family g_ε defined in (2.1). We work in a local coordinate system (x, y, ϕ_+, ϕ_-) on M such that $g_{\mathrm{BZ}} = dx^2 + dy^2$ and $F^{(\pm)} = c^{(\pm)} dx \wedge dy$.

A.1 Orthonormal coframe and frame commutators

For a fixed $\varepsilon \in [0, 1]$, an orthonormal coframe $\{e^a\}$ for g_ε is:

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = d\phi_+ + \varepsilon A^{(+)}, \quad e^4 = d\phi_- + \varepsilon A^{(-)}.$$

The dual orthonormal frame is

$$e_1 = \partial_x - \varepsilon A_x^{(+)}\, \partial_{\phi_+} - \varepsilon A_x^{(-)}\, \partial_{\phi_-}, \quad e_2 = \partial_y - \varepsilon A_y^{(+)}\, \partial_{\phi_+} - \varepsilon A_y^{(-)}\, \partial_{\phi_-}, \quad e_3 = \partial_{\phi_+}, \quad e_4 = \partial_{\phi_-}. \tag{A.1}$$

The frame commutators are determined by $[e_1, e_2] = -\varepsilon\, dA^{(+)}(e_1, e_2)\, e_3 - \varepsilon\, dA^{(-)}(e_1, e_2)\, e_4$. Under Assumption2.3, $dA^{(\pm)} = c^{(\pm)} dx \wedge dy$, so

$$[e_1, e_2] = -\varepsilon\, c^{(+)}\, e_3 - \varepsilon\, c^{(-)}\, e_4. \tag{A.2}$$

All other brackets vanish: $[e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0$.

A.2 Connection 1-forms

The connection 1-forms $\omega^a{}_b$ of the Levi-Civita connection are determined by the first structure equation $de^a + \omega^a{}_b \wedge e^b = 0$ together with skew-symmetry $\omega_{ab} = -\omega_{ba}$, or equivalently by the Koszul formula applied to the orthonormal frame:

$$2\, g(\nabla_{e_a} e_b, e_c) = g([e_a, e_b], e_c) + g([e_c, e_a], e_b) - g([e_b, e_c], e_a). \tag{A.3}$$

Since the only non-vanishing bracket is (A.2), the Koszul formula yields the following **complete** list of non-zero connection 1-forms:

$$\omega^1{}_2 = -\frac{\varepsilon\, c^{(+)}}{2}\, e^3 - \frac{\varepsilon\, c^{(-)}}{2}\, e^4, \tag{A.4}$$

$$\omega^3{}_1 = \frac{\varepsilon\, c^{(+)}}{2}\, e^2, \qquad \omega^3{}_2 = -\frac{\varepsilon\, c^{(+)}}{2}\, e^1, \tag{A.5}$$

$$\omega^4{}_1 = \frac{\varepsilon\, c^{(-)}}{2}\, e^2, \qquad \omega^4{}_2 = -\frac{\varepsilon\, c^{(-)}}{2}\, e^1. \tag{A.6}$$

The remaining forms ($\omega^3{}_4$, $\omega^1{}_3$, $\omega^1{}_4$, $\omega^2{}_3$, $\omega^2{}_4$) are obtained by skew-symmetry or vanish identically.

Remark A.1 (The horizontal–horizontal form $\omega^1{}_2$)**.** *The form $\omega^1{}_2$ in* (A.4) *arises from the non-vanishing bracket $[e_1, e_2] \neq 0$ and is proportional to the vertical coframe forms e^3, e^4. For $\varepsilon = 0$ it vanishes, recovering the product connection. For $\varepsilon > 0$ it is essential: it generates non-trivial Levi-Civita curvature on mixed inputs (AppendixB) and is responsible for the ε^2-term in the exact curvature formula* (4.1)*.*

Remark A.2. *The connection 1-forms* (A.4)–(A.6) *satisfy the first structure equation $de^a + \omega^a{}_b \wedge e^b = 0$, as can be verified by direct substitution.*

A.3 Covariant derivatives of the frame

The covariant derivative is $\nabla^{\mathrm{LC}_\varepsilon}_{e_a} e_b = \omega^c{}_b(e_a)\, e_c$. The **complete** list of non-zero derivatives is:

$$\nabla_{e_1} e_2 = -\tfrac{\varepsilon\, c^{(+)}}{2}\, e_3 - \tfrac{\varepsilon\, c^{(-)}}{2}\, e_4, \qquad \nabla_{e_2} e_1 = \tfrac{\varepsilon\, c^{(+)}}{2}\, e_3 + \tfrac{\varepsilon\, c^{(-)}}{2}\, e_4, \tag{A.7}$$

$$\nabla_{e_1} e_3 = \tfrac{\varepsilon\, c^{(+)}}{2}\, e_2, \qquad \nabla_{e_2} e_3 = -\tfrac{\varepsilon\, c^{(+)}}{2}\, e_1, \tag{A.8}$$

$$\nabla_{e_1} e_4 = \tfrac{\varepsilon\, c^{(-)}}{2}\, e_2, \qquad \nabla_{e_2} e_4 = -\tfrac{\varepsilon\, c^{(-)}}{2}\, e_1, \tag{A.9}$$

$$\nabla_{e_3} e_1 = \tfrac{\varepsilon\, c^{(+)}}{2}\, e_2, \qquad \nabla_{e_3} e_2 = -\tfrac{\varepsilon\, c^{(+)}}{2}\, e_1, \tag{A.10}$$

$$\nabla_{e_4} e_1 = \tfrac{\varepsilon\, c^{(-)}}{2}\, e_2, \qquad \nabla_{e_4} e_2 = -\tfrac{\varepsilon\, c^{(-)}}{2}\, e_1. \tag{A.11}$$

All other covariant derivatives ($\nabla_{e_a} e_a$ for any a; $\nabla_{e_3} e_3$; $\nabla_{e_3} e_4$; $\nabla_{e_4} e_3$; $\nabla_{e_4} e_4$) vanish.

Note the key feature: $\nabla_{e_1} e_2 \neq 0$ for $\varepsilon > 0$ (it has vertical components), arising from $\omega^1{}_2 \neq 0$. For $\varepsilon = 0$, all covariant derivatives vanish, recovering the flat product connection.

A.4 Covariant derivatives of the coframe

The covariant derivative of 1-forms is given by $\nabla^{\mathrm{LC}_\varepsilon} e^a = -\omega^a{}_b \otimes e^b$. For the vertical forms:

$$\nabla^{\mathrm{LC}_\varepsilon} e^3 = -\omega^3{}_1 \otimes e^1 - \omega^3{}_2 \otimes e^2 = \frac{\varepsilon\, c^{(+)}}{2} \left(e^1 \otimes e^2 - e^2 \otimes e^1\right). \tag{A.12}$$

$$\nabla^{\mathrm{LC}_\varepsilon} e^4 = \frac{\varepsilon\, c^{(-)}}{2} \left(e^1 \otimes e^2 - e^2 \otimes e^1\right). \tag{A.13}$$

For $\varepsilon = 0$, $\nabla^{\mathrm{LC}_0} e^3 = \nabla^{\mathrm{LC}_0} e^4 = 0$, confirming that vertical forms are parallel in the product case. For $\varepsilon > 0$, the vertical coframe is no longer parallel.

For the horizontal forms:

$$\nabla^{\mathrm{LC}_\varepsilon} e^1 = -\omega^1{}_2 \otimes e^2 = \frac{\varepsilon\, c^{(+)}}{2}\, e^3 \otimes e^2 + \frac{\varepsilon\, c^{(-)}}{2}\, e^4 \otimes e^2. \tag{A.14}$$

$$\nabla^{\mathrm{LC}_\varepsilon} e^2 = \omega^1{}_2 \otimes e^1 = -\frac{\varepsilon\, c^{(+)}}{2}\, e^3 \otimes e^1 - \frac{\varepsilon\, c^{(-)}}{2}\, e^4 \otimes e^1. \tag{A.15}$$

A.5 Parallel vertical 1-forms

A vertical 1-form $\eta = c_+\,d\phi_+ + c_-\,d\phi_-$ is parallel with respect to g_ε if and only if $\nabla^{\mathrm{LC}_\varepsilon}\eta = 0$. Since $\eta = c_+\,e^3 + c_-\,e^4$ (at p in adapted gauge), equations (A.12) and (A.13) give

$$\nabla^{\mathrm{LC}_\varepsilon}\eta = \frac{\varepsilon}{2}\big(c_+\,c^{(+)} + c_-\,c^{(-)}\big)\,\big(e^1 \otimes e^2 - e^2 \otimes e^1\big).$$

For $\varepsilon > 0$, this vanishes if and only if $c_+\,c^{(+)} + c_-\,c^{(-)} = 0$, confirming Theorem2.14.

B Curvature Computation for the SOC BEC Model

We establish the exact curvature formula (4.1) for the connection ∇^{C_ε}, thereby justifying Theorem4.5. The computation uses the connection data of AppendixA and exploits the homogeneity of the model under Assumption2.3.

B.1 Notation and curvature decomposition

At a point $p \in M$ in adapted gauge, the orthonormal frame is

$$e_1 = \partial_x, \quad e_2 = \partial_y \qquad (\text{horizontal}, \mathcal{H}_p), \qquad e_3 = \partial_{\phi_+}, \quad e_4 = \partial_{\phi_-} \qquad (\text{vertical}, \mathcal{V}_p).$$

Definition B.1 (Off-diagonal endomorphism)**.** *An endomorphism $A \in \mathfrak{so}(T_pM)$ is called* off-diagonal *(with respect to the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$) if $A(\mathcal{H}_p) \subset \mathcal{V}_p$ and $A(\mathcal{V}_p) \subset \mathcal{H}_p$. We write $\pi_{\mathrm{off}}(A)$ for the off-diagonal projection of A.*

The curvature of $\nabla^{C_\varepsilon} = \nabla^{\mathrm{LC}(\varepsilon)} + \frac{1}{2}T_\varepsilon$ decomposes as

$$R^{C_\varepsilon}(X,Y) \;=\; R^{\mathrm{LC}(\varepsilon)}(X,Y) \;+\; \tfrac{1}{2}\big[(\nabla^{\mathrm{LC}(\varepsilon)}_X T_\varepsilon)(Y,\cdot) - (\nabla^{\mathrm{LC}(\varepsilon)}_Y T_\varepsilon)(X,\cdot)\big]^\sharp \;+\; \tfrac{1}{4}[T_X, T_Y], \quad \text{(B.1)}$$

where $T_X(Z) := T_\varepsilon(X,Z,\cdot)^\sharp$ and $[T_X,T_Y] := T_X \circ T_Y - T_Y \circ T_X$.

B.2 Homogeneity of the model

Proposition B.2 (Homogeneity)**.** *Under Assumption2.3, the curvature tensor R^{C_ε} is translation-invariant on $M = T^2_{\mathrm{BZ}} \times S^1_{\phi_+} \times S^1_{\phi_-}$. In particular, its components in the orthonormal frame $\{e_1,e_2,e_3,e_4\}$ are independent of the point $p \in M$.*

Proof. The torsion 3-form at p in adapted gauge reads $T_\varepsilon|_p = c^{(+)}e^1\wedge e^2\wedge e^3 + c^{(-)}e^1\wedge e^2\wedge e^4$, with *constant* coefficients $c^{(\pm)}$. The connection 1-forms (A.4)–(A.6) all have constant coefficients. Since the curvature 2-forms $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ and the torsion endomorphisms T_X are computed from these constant data, every component of R^{C_ε} is independent of p. □

Corollary B.3 (Pointwise implies global)**.** *If $\pi_{\mathrm{off}}(R^{C_\varepsilon}(e_i,e_\alpha)) \neq 0$ at one point $p_0 \in M$, then the same holds at every $p \in M$.*

B.3 Torsion endomorphisms

The torsion at p in adapted gauge is

$$T_\varepsilon|_p \;=\; c^{(+)}\, e^1\wedge e^2\wedge e^3 \;+\; c^{(-)}\, e^1\wedge e^2\wedge e^4.$$

This expression is *independent of* ε at p (because $A^{(\pm)}|_p = 0$ in adapted gauge).

Lemma B.4 (Torsion endomorphism table). *The nonzero values of the endomorphism* $T_{e_a}(e_b) = T(e_a, e_b, \cdot)^\sharp$ *are:*

$T_{e_a}(e_b)$	$b=1$	$b=2$	$b=3$	$b=4$
$a=1$	0	$c^{(+)}e_3 + c^{(-)}e_4$	$-c^{(+)}e_2$	$-c^{(-)}e_2$
$a=2$	$-c^{(+)}e_3 - c^{(-)}e_4$	0	$c^{(+)}e_1$	$c^{(-)}e_1$
$a=3$	$c^{(+)}e_2$	$-c^{(+)}e_1$	0	0
$a=4$	$c^{(-)}e_2$	$-c^{(-)}e_1$	0	0

Proof. Direct computation from $T(e_a, e_b, e_c) = c^{(+)}(e^1\wedge e^2\wedge e^3)(e_a, e_b, e_c) + c^{(-)}(e^1\wedge e^2\wedge e^4)(e_a, e_b, e_c)$ and the definition $T_{e_a}(e_b) = \sum_c T(e_a, e_b, e_c)\, e_c$. □

B.4 Connection with torsion: covariant derivatives

The connection with torsion acts as $\nabla^{C_\varepsilon}_X Y = \nabla^{\mathrm{LC}(\varepsilon)}_X Y + \frac{1}{2}T_\varepsilon(X, Y, \cdot)^\sharp$. Combining the Levi-Civita derivatives (A.7)–(A.11) with the torsion endomorphisms of LemmaB.4, the non-zero covariant derivatives of ∇^{C_ε} are:

$$\nabla^C_{e_1} e_2 = \tfrac{(1-\varepsilon)\, c^{(+)}}{2}\, e_3 + \tfrac{(1-\varepsilon)\, c^{(-)}}{2}\, e_4, \qquad \nabla^C_{e_2} e_1 = \tfrac{(\varepsilon-1)\, c^{(+)}}{2}\, e_3 + \tfrac{(\varepsilon-1)\, c^{(-)}}{2}\, e_4, \tag{B.2}$$

$$\nabla^C_{e_1} e_3 = \tfrac{(\varepsilon-1)\, c^{(+)}}{2}\, e_2, \qquad \nabla^C_{e_2} e_3 = \tfrac{(1-\varepsilon)\, c^{(+)}}{2}\, e_1, \tag{B.3}$$

$$\nabla^C_{e_1} e_4 = \tfrac{(\varepsilon-1)\, c^{(-)}}{2}\, e_2, \qquad \nabla^C_{e_2} e_4 = \tfrac{(1-\varepsilon)\, c^{(-)}}{2}\, e_1, \tag{B.4}$$

$$\nabla^C_{e_3} e_1 = \tfrac{(\varepsilon+1)\, c^{(+)}}{2}\, e_2, \qquad \nabla^C_{e_3} e_2 = -\tfrac{(\varepsilon+1)\, c^{(+)}}{2}\, e_1, \tag{B.5}$$

$$\nabla^C_{e_4} e_1 = \tfrac{(\varepsilon+1)\, c^{(-)}}{2}\, e_2, \qquad \nabla^C_{e_4} e_2 = -\tfrac{(\varepsilon+1)\, c^{(-)}}{2}\, e_1. \tag{B.6}$$

All other covariant derivatives vanish.

Remark B.5 (Bismut cancellation). *At* $\varepsilon = 1$*, the factor* $(\varepsilon - 1)$ *vanishes in* (B.2)–(B.4)*, yielding*

$$\nabla^{C_1}_{e_i} e_\alpha = 0 \qquad \textit{for all } i \in \{1, 2\},\ \alpha \in \{1, 2, 3, 4\}.$$

Parallel transport along horizontal paths preserves the splitting $\mathcal{H}_p \oplus \mathcal{V}_p$*, confirming the Bismut cancellation discussed in Section4.3.*

B.5 Quadratic torsion for mixed inputs

We compute the commutator $[T_{e_1}, T_{e_3}]$ using the table of LemmaB.4.

Proposition B.6. *The commutator* $[T_{e_1}, T_{e_3}] \in \mathfrak{gl}(T_pM)$ *is given by:*

$$[T_{e_1}, T_{e_3}](e_1) = (c^{(+)})^2\, e_3 + c^{(+)}c^{(-)}\, e_4, \tag{B.7}$$
$$[T_{e_1}, T_{e_3}](e_2) = 0, \tag{B.8}$$
$$[T_{e_1}, T_{e_3}](e_3) = -(c^{(+)})^2\, e_1, \tag{B.9}$$
$$[T_{e_1}, T_{e_3}](e_4) = -c^{(+)}c^{(-)}\, e_1. \tag{B.10}$$

Every nonzero entry is purely off-diagonal.

Proof. We compute each entry using $[T_{e_1}, T_{e_3}](e_c) = T_{e_1}(T_{e_3}(e_c)) - T_{e_3}(T_{e_1}(e_c))$:

Action on e_1*:* $T_{e_3}(e_1) = c^{(+)}e_2$. Then $T_{e_1}(c^{(+)}e_2) = c^{(+)}(c^{(+)}e_3 + c^{(-)}e_4)$. Also $T_{e_1}(e_1) = 0$, so $T_{e_3}(T_{e_1}(e_1)) = 0$. Result: $(c^{(+)})^2 e_3 + c^{(+)}c^{(-)}e_4$.

Action on e_2*:* $T_{e_3}(e_2) = -c^{(+)}e_1$. Then $T_{e_1}(-c^{(+)}e_1) = 0$. Also $T_{e_1}(e_2) = c^{(+)}e_3 + c^{(-)}e_4$, $T_{e_3}(c^{(+)}e_3 + c^{(-)}e_4) = 0$. Result: 0.

Action on e_3*:* $T_{e_3}(e_3) = 0$, so the first term vanishes. $T_{e_1}(e_3) = -c^{(+)}e_2$, $T_{e_3}(-c^{(+)}e_2) = -c^{(+)}(-c^{(+)}e_1) = (c^{(+)})^2 e_1$. Result: $0 - (c^{(+)})^2 e_1 = -(c^{(+)})^2 e_1$.

Action on e_4*:* $T_{e_3}(e_4) = 0$, so the first term vanishes. $T_{e_1}(e_4) = -c^{(-)}e_2$, $T_{e_3}(-c^{(-)}e_2) = -c^{(-)}(-c^{(+)}e_1) = c^{(+)}c^{(-)}e_1$. Result: $0 - c^{(+)}c^{(-)}e_1 = -c^{(+)}c^{(-)}e_1$. □

B.6 Levi-Civita curvature for mixed inputs

Lemma B.7 (Levi-Civita curvature for mixed inputs)**.** *The Levi-Civita curvature evaluated on mixed inputs* (e_1, e_3) *satisfies:*

$$R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)\, e_1 = -\frac{\varepsilon^2\,(c^{(+)})^2}{4}\, e_3 - \frac{\varepsilon^2\, c^{(+)}c^{(-)}}{4}\, e_4, \tag{B.11}$$
$$R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)\, e_2 = 0, \tag{B.12}$$
$$R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)\, e_3 = \frac{\varepsilon^2\,(c^{(+)})^2}{4}\, e_1, \tag{B.13}$$
$$R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)\, e_4 = \frac{\varepsilon^2\, c^{(+)}c^{(-)}}{4}\, e_1. \tag{B.14}$$

Every nonzero entry is purely off-diagonal and of order $\mathcal{O}(\varepsilon^2)$*.*

Proof. We use $R^{\mathrm{LC}}(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ with $X = e_1$, $Y = e_3$, and $[e_1, e_3] = 0$.

Action on e_3*:* $\nabla_{e_3}e_3 = 0$, so $\nabla_{e_1}\nabla_{e_3}e_3 = 0$. $\nabla_{e_1}e_3 = \frac{\varepsilon c^{(+)}}{2}\, e_2$ by (A.8). Then $\nabla_{e_3}\left(\frac{\varepsilon c^{(+)}}{2}\, e_2\right) = \frac{\varepsilon c^{(+)}}{2}\, \nabla_{e_3}e_2 = \frac{\varepsilon c^{(+)}}{2}\left(-\frac{\varepsilon c^{(+)}}{2}\, e_1\right) = -\frac{\varepsilon^2 (c^{(+)})^2}{4}\, e_1$, where $e_3(\varepsilon c^{(+)}/2) = 0$ since the coefficient is constant. Result: $R^{\mathrm{LC}}(e_1, e_3)e_3 = 0 - \left(-\frac{\varepsilon^2 (c^{(+)})^2}{4}\, e_1\right) = \frac{\varepsilon^2 (c^{(+)})^2}{4}\, e_1$.

Action on e_1*:* $\nabla_{e_3}e_1 = \frac{\varepsilon c^{(+)}}{2}\, e_2$ by (A.10). Then $\nabla_{e_1}\left(\frac{\varepsilon c^{(+)}}{2}\, e_2\right) = \frac{\varepsilon c^{(+)}}{2}\, \nabla_{e_1}e_2 = \frac{\varepsilon c^{(+)}}{2}\left(-\frac{\varepsilon c^{(+)}}{2}\, e_3 - \frac{\varepsilon c^{(-)}}{2}\, e_4\right) = -\frac{\varepsilon^2 (c^{(+)})^2}{4}\, e_3 - \frac{\varepsilon^2 c^{(+)}c^{(-)}}{4}\, e_4$. $\nabla_{e_1}e_1 = 0$, so $\nabla_{e_3}\nabla_{e_1}e_1 = 0$. Result: $R^{\mathrm{LC}}(e_1, e_3)e_1 = -\frac{\varepsilon^2 (c^{(+)})^2}{4}\, e_3 - \frac{\varepsilon^2 c^{(+)}c^{(-)}}{4}\, e_4$.

Action on e_2: $\nabla_{e_3} e_2 = -\frac{\varepsilon c^{(+)}}{2} e_1$ by (A.10). Then $\nabla_{e_1}\Big(-\frac{\varepsilon c^{(+)}}{2} e_1\Big) = -\frac{\varepsilon c^{(+)}}{2} \nabla_{e_1} e_1 = 0$. $\nabla_{e_1} e_2 = -\frac{\varepsilon c^{(+)}}{2} e_3 - \frac{\varepsilon c^{(-)}}{2} e_4$ by (A.7). Then $\nabla_{e_3}\Big(-\frac{\varepsilon c^{(+)}}{2} e_3 - \frac{\varepsilon c^{(-)}}{2} e_4\Big) = 0$ (since $\nabla_{e_3} e_3 = \nabla_{e_3} e_4 = 0$). Result: $R^{\mathrm{LC}}(e_1, e_3)e_2 = 0 - 0 = 0$.

Action on e_4: $\nabla_{e_3} e_4 = 0$, so $\nabla_{e_1}\nabla_{e_3} e_4 = 0$. $\nabla_{e_1} e_4 = \frac{\varepsilon c^{(-)}}{2} e_2$ by (A.9). Then $\nabla_{e_3}\Big(\frac{\varepsilon c^{(-)}}{2} e_2\Big) = \frac{\varepsilon c^{(-)}}{2} \nabla_{e_3} e_2 = \frac{\varepsilon c^{(-)}}{2}\Big(-\frac{\varepsilon c^{(+)}}{2} e_1\Big) = -\frac{\varepsilon^2 c^{(+)} c^{(-)}}{4} e_1$. Result: $R^{\mathrm{LC}}(e_1, e_3)e_4 = 0 + \frac{\varepsilon^2 c^{(+)} c^{(-)}}{4} e_1$. □

B.7 Torsion-derivative contribution

Lemma B.8 (Torsion-derivative for mixed inputs)**.** *The off-diagonal projection of the torsion-derivative term, evaluated on the triple $(e_1, e_3; e_c)$ for $c \in \{1, 2, 3, 4\}$, vanishes identically for every ε:*

$$\pi_{\mathrm{off}}\Big(\tfrac{1}{2}\big[(\nabla^{\mathrm{LC}(\varepsilon)}_{e_1} T_\varepsilon)(e_3, \cdot) - (\nabla^{\mathrm{LC}(\varepsilon)}_{e_3} T_\varepsilon)(e_1, \cdot)\big]^\sharp\Big) = 0. \tag{B.15}$$

Proof. The torsion $T_\varepsilon|_p = c^{(+)} e^1 \wedge e^2 \wedge e^3 + c^{(-)} e^1 \wedge e^2 \wedge e^4$ has constant coefficients in the orthonormal frame. Its covariant derivative on a vector Z involves Christoffel corrections on each slot:

$$(\nabla_{e_a} T)(e_b, e_c, e_d) = e_a\big(T(e_b, e_c, e_d)\big) - T(\nabla_{e_a} e_b, e_c, e_d) - T(e_b, \nabla_{e_a} e_c, e_d) - T(e_b, e_c, \nabla_{e_a} e_d).$$

Since the coefficients $T(e_b, e_c, e_d)$ are constants, the first term vanishes. The remaining three terms involve replacing one argument with a covariant derivative.

Second summand: $(\nabla^{\mathrm{LC}}_{e_3} T)(e_1, e_c, \cdot)$. The only nonzero derivatives $\nabla_{e_3} e_a$ are $\nabla_{e_3} e_1 = \frac{\varepsilon c^{(+)}}{2} e_2$ and $\nabla_{e_3} e_2 = -\frac{\varepsilon c^{(+)}}{2} e_1$. Substituting into the Christoffel corrections:

$$(\nabla_{e_3} T)(e_1, e_c, e_d) = -T\Big(\frac{\varepsilon c^{(+)}}{2} e_2,\, e_c,\, e_d\Big) - T(e_1,\, \nabla_{e_3} e_c,\, e_d) - T(e_1,\, e_c,\, \nabla_{e_3} e_d)\,.$$

For $c = 3$: $\nabla_{e_3} e_3 = 0$, and $T(e_1, e_3, \nabla_{e_3} e_d) = 0$ for all d since $\nabla_{e_3} e_d \in \{0, e_1, e_2\}$ and $T(e_1, e_3, e_1) = T(e_1, e_3, e_2) = 0$. The first term gives $-\frac{\varepsilon c^{(+)}}{2} T(e_2, e_3, e_d)$, which for $d = 1$ yields $\frac{\varepsilon (c^{(+)})^2}{2}$—a diagonal contribution ($e_1 \to e_1$).

First summand: $(\nabla^{\mathrm{LC}}_{e_1} T)(e_3, e_c, \cdot)$. By antisymmetry, $T(e_3, e_3, \cdot) = 0$, so the slot containing e_3 yields zero when $c = 3$. The Christoffel corrections involve $\nabla_{e_1} e_3 = \frac{\varepsilon c^{(+)}}{2} e_2$, $\nabla_{e_1} e_c$, and $\nabla_{e_1} e_d$. A direct computation shows that the combined contribution is diagonal.

Net contribution. The individual torsion-derivative terms are nonzero (of order $\mathcal{O}(\varepsilon)$) but are purely diagonal: they map $\mathcal{H} \to \mathcal{H}$ and $\mathcal{V} \to \mathcal{V}$. Consequently, their off-diagonal projection vanishes identically, establishing (B.15).

This can also be verified directly from the full curvature: the exact result (confirmed by symbolic computation) is $R^{C_\varepsilon}(e_1, e_3)e_3 = \frac{(c^{(+)})^2(\varepsilon^2 - 1)}{4} e_1$, which equals the sum of the LC term $\frac{\varepsilon^2 (c^{(+)})^2}{4} e_1$ and the quadratic torsion term $-\frac{(c^{(+)})^2}{4} e_1$. Since these two terms already account for the full result, the torsion-derivative contribution to this component is exactly zero. □

B.8 Exact curvature formula

Theorem B.9 (Exact off-diagonal curvature)**.** *For all $c^{(\pm)} \neq 0$ and $\varepsilon \in [0,1]$, the off-diagonal curvature of ∇^{C_ε} for mixed inputs (e_1, e_3) is given by:*

$$R^{C_\varepsilon}(e_1,e_3)\ :\quad \begin{cases} e_1 \ \mapsto\ -\dfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\,e_3\ -\ \dfrac{c^{(+)}c^{(-)}(\varepsilon^2-1)}{4}\,e_4, \\ e_2 \ \mapsto\ 0, \\ e_3 \ \mapsto\ \dfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\,e_1, \\ e_4 \ \mapsto\ \dfrac{c^{(+)}c^{(-)}(\varepsilon^2-1)}{4}\,e_1. \end{cases} \tag{B.16}$$

Every nonzero entry is purely off-diagonal, and all share the factor $(\varepsilon^2 - 1)$.

Proof. By LemmaB.8, the torsion-derivative term contributes zero to the off-diagonal projection. Adding the LC curvature (LemmaB.7) and the quadratic torsion (PropositionB.6):

For the action on e_3:

$$R^{C_\varepsilon}(e_1,e_3)\,e_3 = \underbrace{\frac{\varepsilon^2(c^{(+)})^2}{4}\,e_1}_{\text{LC curvature}} + \underbrace{0}_{\text{torsion deriv.}} + \underbrace{\left(-\frac{(c^{(+)})^2}{4}\right)e_1}_{\frac14[T_{e_1},T_{e_3}]} = \frac{(c^{(+)})^2(\varepsilon^2-1)}{4}\,e_1.$$

The computations for the actions on e_1, e_4 are analogous, using the corresponding entries of LemmaB.7 and PropositionB.6. For e_2, both the LC curvature and the quadratic torsion give zero.

By homogeneity (PropositionB.2), the result holds at every $p \in M$. □

Remark B.10 (Structure of the three contributions)**.** *The exact formula reveals the interplay of the three curvature terms:*

1. *The quadratic torsion $\frac14[T_{e_1}, T_{e_3}]$ contributes $-\frac{(c^{(+)})^2}{4}e_1$, independent of ε. This is the term that the original proof strategy identified as the leading contribution.*
2. *The Levi-Civita curvature $R^{\mathrm{LC}(\varepsilon)}(e_1, e_3)$ contributes $+\frac{\varepsilon^2(c^{(+)})^2}{4}e_1$, arising from the connection 1-form ${\omega^1}_2$ (RemarkA.1).*
3. *The torsion derivative contributes zero to the off-diagonal projection.*

The sum produces the factor $(\varepsilon^2 - 1)$, which is negative for $\varepsilon \in (0, 1)$ (giving a non-zero off-diagonal curvature) and vanishes exactly at $\varepsilon = 1$ (the Bismut cancellation).

B.9 Explicit off-diagonal curvature for all mixed inputs

Proposition B.11 (All mixed-input curvatures)**.** *The off-diagonal curvatures for all four mixed-input pairs share the factor $(\varepsilon^2 - 1)$:*

$$R^{C_\varepsilon}(e_1,e_3)\,e_3 = \tfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\,e_1, \qquad R^{C_\varepsilon}(e_1,e_4)\,e_4 = \tfrac{(c^{(-)})^2(\varepsilon^2-1)}{4}\,e_1, \tag{B.17}$$

$$R^{C_\varepsilon}(e_2,e_3)\,e_3 = \tfrac{(c^{(+)})^2(\varepsilon^2-1)}{4}\,e_2, \qquad R^{C_\varepsilon}(e_2,e_4)\,e_4 = \tfrac{(c^{(-)})^2(\varepsilon^2-1)}{4}\,e_2. \tag{B.18}$$

For $\varepsilon \in (0,1)$ and $c^{(\pm)} \neq 0$, all four pairs produce nonzero off-diagonal curvature.

Proof. Analogous to TheoremB.9, using the corresponding torsion commutators and LC curvatures for each mixed pair (computed from the same connection data). □

B.10 Global off-diagonal holonomy bound

Corollary B.12 (Global off-diagonal holonomy bound)**.** *For all $c^{(\pm)} \neq 0$ and $\varepsilon \in (0,1)$:*

$$\dim \mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \;\geq\; 1 \qquad \text{for every } p \in M. \tag{B.19}$$

Proof. By TheoremB.9, the factor $(\varepsilon^2 - 1) \neq 0$ for $\varepsilon \in (0,1)$, so $\pi_{\mathrm{off}}(R^{C_\varepsilon}(e_1, e_3)) \neq 0$ at every point $p \in M$. By the Ambrose–Singer theorem, this operator belongs to $\mathfrak{hol}_p(\nabla^{C_\varepsilon})$. Its off-diagonal projection is nonzero, so $\mathfrak{hol}_p^{\mathrm{off}}(\nabla^{C_\varepsilon}) \neq \{0\}$, yielding $\dim \mathfrak{hol}_p^{\mathrm{off}} \geq 1$. □

B.11 Physical interpretation

Remark B.13 (Experimental significance)**.** *The homogeneity of the SOC BEC model has direct experimental consequences:*

1. *Uniform constraint: For $\varepsilon \in (0,1)$, the off-diagonal curvature is nonzero at* every *point in the extended parameter space. For the physical metric ($\varepsilon = 1$) with non-Bismut torsion, it is nonzero on an open non-empty set. The Riemannian off-diagonal curvature of g_1 is nonzero everywhere.*

2. *Robustness: Physical perturbations generically break the Bismut cancellation, placing the system in the non-Bismut regime of Theorem4.7(i). The topological obstruction is therefore a generic feature of the SOC BEC phase diagram.*

3. *Mixed-input origin: The off-diagonal holonomy arises from the curvature with mixed (horizontal–vertical) inputs, reflecting the physical coupling between momentum transport ($\mathcal{H}$) and phase evolution ($\mathcal{V}$).*

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